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SEMISTABLE TYPES OF HYPERELLIPTIC CURVES

TIM DOKCHITSER, VLADIMIR DOKCHITSER, CÉLINE MAISTRET, ADAM MORGAN

ABSTRACT. In this paper, we explore three combinatorial descriptions of semistable types of hyperelliptic curves over local fields: dual graphs, their quotient trees by the hyperelliptic involution, and configurations of the roots of the defining equation (‘cluster pictures’). We construct explicit combinatorial one-to-one correspondences between the three, which furthermore respect automorphisms and allow to keep track of the monodromy pairing and the Tamagawa group of the Jacobian. We introduce a classification scheme and a naming convention for semistable types of hyperelliptic curves and types with a Frobenius action. This is the higher genus analogue of the distinction between good, split and non-split multiplicative reduction for elliptic curves. Our motivation is to understand L -factors, Galois representations, conductors, Tamagawa numbers and other local invariants of hyperelliptic curves and their Jacobians.

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Key words and phrases. Hyperelliptic curves, hyperelliptic graphs, BY trees, cluster pictures, Tamagawa group, semistable reduction.

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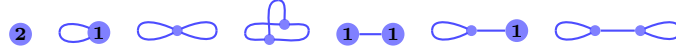
1. INTRODUCTION

Suppose K is a field with a discrete valuation, say of odd residue characteristic, and C/K is a hyperelliptic curve of genus g ,

$$C : y^2 = f(x).$$

Our motivation is the study of the arithmetic of C and its Jacobian, including its minimal model, Tamagawa number, L -factor, conductor and other invariants related to the Birch–Swinnerton-Dyer conjecture. It would be desirable to have a classification of reduction types in the fashion of Kodaira types for elliptic curves, which moreover would take into consideration non-algebraically closed residue fields. In order to do so, for semistable curves, this paper develops a correspondence between three natural combinatorial objects attached to C that control its arithmetic. The correspondence is explicit and gives a simple way to pass between these objects in practice.

1.1. Correspondence. First, C has semistable reduction over some finite extension F/K : it has a model over the ring of integers of F with stable special fibre \bar{C} . Thus, \bar{C} has only ordinary double points as singularities, and $\text{Aut}(\bar{C})$ is finite (assuming $g \geq 2$ for the moment). Associated to \bar{C} is its dual graph G , with a vertex for each geometric irreducible component, decorated with its genus and an edge for each intersection. It is often referred to as a ‘semistable type’ of C ; e.g., in genus 2 there are seven types (omitting genus 0 markings):



Second, G has an involution ι that comes from the hyperelliptic involution $y \mapsto -y$ on C , and the topological quotient $G/\langle \iota \rangle$ is a tree, say T . It has genus markings on the vertices as well, and a natural 2-colouring: colour points over which $G \rightarrow T$ is 2:1 yellow, and the branch locus blue. In genus 2, the corresponding trees are



Third, the set $X \subset \bar{K}$ of the $2g+1$ or $2g+2$ roots of the defining polynomial $f(x)$ gives another natural combinatorial invariant — how the roots ‘cluster’ together. Call a non-empty subset $\mathfrak{s} \subset X$ a *cluster* if it is of the form $X \cap (\text{some disc in } \bar{K})$, and view X abstractly as a finite set with a collection Σ of clusters $\mathfrak{s} \subset X$, a *cluster picture*. Different presentations $y^2 = f(x)$ of the same curve may give different cluster pictures; however there is an equivalence relation induced by Möbius transformations of the roots. When $|X| = 6$ ($g = 2$), there are seven equivalence classes, represented by



The leftmost one illustrates the 6 roots being all equidistant, in the next one the last two roots are closer to each other than to the other four, and so on.

The three sets of 7 pictures raise the obvious question, to which the answer turns out to be ‘Yes’. There is an established combinatorial notion of a ‘hyperelliptic graph’. In this paper, we introduce ‘BY (blue/yellow) trees’ and ‘cluster pictures’, again in a combinatorial fashion, and formally define genus and equivalence. We then prove

Theorem 1.1 (Main correspondence). *There is an explicit genus-preserving one-to-one correspondence between*

- *Hyperelliptic graphs up to isomorphism,*
- *BY trees up to isomorphism,*
- *Cluster pictures up to equivalence.*



In order to work with a fixed model of a hyperelliptic curve, it is also natural to ask for a graph-theoretic counterpart of a cluster picture that determines it up to isomorphism, rather than up to equivalence. The right notions turn out to be *open BY tree* / *open hyperelliptic graph*, with one extra edge or ι -orbit of edges with no endpoint (‘going off to ∞ ’). An open hyperelliptic graph G has a unique largest hyperelliptic subgraph, its *core* \tilde{G} , and G, G' are called *equivalent* if $\tilde{G} \cong \tilde{G}'$; similarly for BY trees T, \tilde{T} .

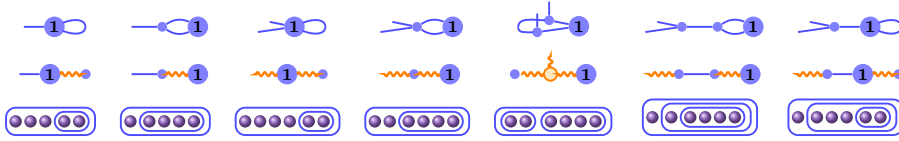
The refined version of Theorem 1.1 is

Theorem 1.2 (Open correspondence). *There is an explicit genus-preserving and equivalence-preserving one-to-one correspondence between*

- *Open hyperelliptic graphs up to isomorphism,*
- *Open BY trees up to isomorphism,*
- *Cluster pictures up to isomorphism.*

The construction is summarized in Table 4.20.

For example, here are open hyperelliptic graphs with core , open BY trees with core , and the corresponding cluster pictures that form one full equivalence class:



1.2. Invariants. The dual graph G of a semistable curve carries several important invariants, notably

- A *metric* (edges have length),
- *Automorphisms* (e.g. coming from the action of Galois),
- Homology *lattice* $\Lambda = H_1(G, \mathbb{Z})$,
- Symmetric positive-definite *pairing*¹ on Λ induced by the metric,

¹the monodromy pairing of Grothendieck [10]

- *Tamagawa group*² $\Phi(G) = \Lambda^\vee / \Lambda$.

All of these have counterparts for BY trees and cluster pictures:

- The analogue of an (open or not) *metric* hyperelliptic graph G / BY tree T (edges have length) is a *metric cluster* picture Σ , with distances between clusters.
- For a BY tree T , the counterpart of $\Lambda_G = H_1(G, \mathbb{Z})$ is the relative homology group $\Lambda_T = H_1(T, T_b, \mathbb{Z})$ with respect to the blue part $T_b \subset T$. For a cluster picture Σ , Λ_Σ is, essentially, a permutation module on certain clusters ('even but not übereven').
- In all three settings, the metric determines a pairing on Λ . When all edge lengths of G are integers, we call G *integral* and this notion also transports to T and Σ as well. In this case, the pairing is \mathbb{Z} -valued, and we can define the *Tamagawa group* $\Phi = \Lambda^\vee / \Lambda$.
- The group $\text{Aut } G$ corresponds to the group $\text{Aut } T$ of automorphisms of T together with a choice of sign for each yellow component, and to the group $\text{Aut } \Sigma$ of permutations of clusters in Σ with a choice of (compatible) signs for clusters of even size.

Theorem 1.3. *The correspondence in Theorem 1.2 extends to the metric case. Suppose G , T and Σ correspond to one another. Choose a section $s : G/\langle \iota \rangle \rightarrow G$. Then there are canonical isomorphisms*

$$\text{Aut}(\tilde{G}) \cong \text{Aut}(\tilde{T}) \cong \text{Aut}(\Sigma)$$

and canonical $\text{Aut}(\cdot)$ -equivariant isomorphisms

$$\Lambda_{\tilde{G}} \cong \Lambda_{\tilde{T}} \cong \Lambda_\Sigma$$

as lattices with a pairing. If G is integral, they induce isomorphisms

$$\Phi(\tilde{G}) \cong \Phi(\tilde{T}) \cong \Phi(\Sigma).$$

The Tamagawa group $\Phi(\tilde{G})$ is also equivariantly isomorphic to the graph-theoretic Jacobian of G , see Proposition 7.10. As one application, we get a simple description of the 2-torsion in $\Phi(\tilde{G})$.

Corollary 1.4. *Let G be an integral hyperelliptic graph of genus ≥ 2 , $G^{\langle \iota \rangle}$ the set of fixed points of the involution ι , and \mathcal{W} the set of those connected components of $G^{\langle \iota \rangle}$ that contain a point of integer distance from a vertex. Then we have isomorphisms of $\text{Aut } G$ -modules*

$$\Phi(G)[2] \cong \begin{cases} 0 & \text{if } \mathcal{W} = \emptyset \text{ and } \text{rk } H_1(G, \mathbb{Z}) \text{ even,} \\ \mathbb{F}_2 & \text{if } \mathcal{W} = \emptyset \text{ and } \text{rk } H_1(G, \mathbb{Z}) \text{ odd,} \\ \ker(\mathbb{F}_2[\mathcal{W}] \xrightarrow{\text{sum}} \mathbb{F}_2) & \text{otherwise.} \end{cases}$$

(Here 'sum' denotes the sum of the coefficients map.)

²also known in graph theory as the Jacobian (see §7.2), Picard group or the sandpile group of G ; for curves over local fields this is the group of connected components of the Néron model of the Jacobian, and the size of its Frobenius invariants is called the Tamagawa number

There is an algorithm due to Betts [1] that computes, for a BY tree T , the Tamagawa group $\Phi(T)$ and the group of invariants $\Phi(T)^{F=1}$ for $F \in \text{Aut } T$. By the theorem above, it gives a way to compute $\Phi(G)$ and $\Phi(\Sigma)$ for hyperelliptic graphs and cluster pictures as well.

1.3. Hyperelliptic curves in odd residue characteristic. The present paper is purely combinatorial, and was motivated by its geometric counterpart [9] that studies the arithmetic and geometry of hyperelliptic curves over local fields. We briefly sketch the results for the interested reader.

Let K be a local field of odd residue characteristic p , with valuation val_K on the separable closure \bar{K} . As before, let C/K be a hyperelliptic curve

$$C : y^2 = f(x)$$

of genus ≥ 2 . Suppose it is semistable over some finite Galois extension F/K . The dual graph G_C of the special fibre of the minimal regular model of C/F is naturally a metric hyperelliptic graph, and it has a Galois action³

$$\text{Gal}(\bar{K}/K) \rightarrow \text{Aut } G_C.$$

The cluster picture $\Sigma_C = (X, \Sigma)$ is given by the collections of roots that are cut out by p -adic discs,

$$X = \{\text{roots of } f \text{ in } \bar{K}\}, \quad \Sigma = \{\mathfrak{s} = X \cap D \mid D \subset \bar{K} \text{ disc, } \mathfrak{s} \neq \emptyset\}.$$

It carries a metric, with the distance between clusters determined by the p -adic distances between the roots:

$$\delta(\mathfrak{s}, \mathfrak{s}') = \text{diameter}(\mathfrak{s}) + \text{diameter}(\mathfrak{s}') - 2 \cdot \text{diameter}(\mathfrak{s} \cup \mathfrak{s}'),$$

where for $X' \subseteq X$, $\text{diameter}(X') = \min_{r, r' \in X'}(\text{val}_K(r - r'))$. An automorphism $\sigma \in \text{Gal}(\bar{K}/K)$ acts on X and permutes the clusters. Moreover, one can assign a natural sign $\epsilon_\sigma(\mathfrak{s}) = \pm 1$ for every even cluster \mathfrak{s} . This gives a homomorphism

$$\text{Gal}(\bar{K}/K) \rightarrow \text{Aut } \Sigma_C.$$

Having produced both the hyperelliptic graph G_C and the cluster picture Σ_C , we can now state

Theorem 1.5 ([9]). *The core of the hyperelliptic graph associated to the cluster picture Σ_C by Theorem 1.2 is isomorphic to the dual graph G_C . The isomorphism preserves the metric and the $\text{Gal}(\bar{K}/K)$ -action.*

Thus, from a cluster picture of C/K , which is an elementary invariant constructed from the roots of $f(x)$, we recover the semistable model of C/F together with the Galois action. This allows us to determine some of the main arithmetic invariants of C , such as

- Necessary and sufficient conditions for C/K to be semistable;
- The Galois representation $H_{\text{ét}}^1(C/\bar{K}, \mathbb{Q}_l)$;
- The conductor of C ;

³This is slightly non-trivial, and relies on uniqueness of the minimal regular model of C over F

- Equations for the minimal regular model of C/F ;
- The Tamagawa number of the Jacobian of C over F .⁴

1.4. Classification of semistable types and naming convention. Recall that one of the main motivations for studying hyperelliptic graphs, BY trees and cluster pictures was to produce a classification of semistable types of hyperelliptic curves. The results in this paper allow one to produce such a classification in any genus; the tables in §9 classify objects up to genus 3 and objects together with an automorphism in genus 1 and 2. Keeping track of an automorphism is important, as the action of Frobenius on the dual graph of the special fiber of the minimal regular model of hyperelliptic curves is key to the study of their arithmetic. One standard application of such classifications is that they enable one to use a systematic case by case analysis. For example, [15] employs the one in the present paper to prove a general result on the parity of ranks of Jacobians of genus 2 curves.

We end by proposing a naming convention for BY trees or, equivalently, hyperelliptic graphs and cluster pictures. In the context of semistable curves of genus 1 or 2, our notation is compatible with that of Kodaira [14] and Namikawa–Ueno [16]. As an illustration, the seven types in genus 2 shown previously get the names

$$2 \quad 1_n \quad I_{n,m} \quad U_{n,m,k} \quad 1 \times 1 \quad 1 \times I_n \quad I_n \times I_m$$

(with the same ordering). Moreover, we also allow an arbitrary automorphism to be encoded in the type. For example, for type 1_n above, there are two choices of automorphisms giving

$$1_n^+ \quad 1_n^-.$$

These correspond to a genus 2 curve whose reduction has one split or non-split node, respectively.

Related work. Hyperelliptic graphs and their link to hyperelliptic curves are well-known [3, 7, 8, 13]. Our definition (3.2) is as in [7, 13]; it is stronger than that of [8] which does not require condition (3).⁵ Cluster pictures (and, to some extent, BY trees) appear implicitly in [6] in setting of rigid geometry. As far as we are aware, neither the correspondence nor an analysis of automorphism groups (which is important when the residue field is not algebraically closed) have been studied.

1.5. Layout. §2 recalls terminology about graphs and properties of their homology, relative homology and the natural pairing coming from edge lengths. §3 introduces hyperelliptic graphs, BY trees and cluster pictures. §4 and §5 state and prove the one-to-one correspondence between hyperelliptic graphs,

⁴Note also that Corollary 1.4 fully describes the 2-torsion of the Tamagawa group.

⁵Condition (3) of 3.2 is needed for geometric reasons, for otherwise there are no hyperelliptic curves with such special fibres; see [13] Thm 1.2 and compare 36 types in genus 3 of [8] with 32 in Table 9.4. From the point of view of our 1-1 correspondence, it is forced automatically by cluster pictures.

BY trees and cluster pictures. §6 shows that it preserves the lattice Λ as a module under automorphism groups. §7 discusses Tamagawa groups of hyperelliptic graphs. The naming convention is addressed in §8. The classification of semistable types is discussed in §9.

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1.6. Notation. Throughout the paper we use the following notation:

| | |
|------------------------|--|
| G | hyperelliptic graph / open hyperelliptic graph (§3.1) |
| T | BY tree / open BY tree (§3.2) |
| Σ | cluster picture (§3.3) |
| \mathfrak{s} | cluster (element of Σ) |
| g | genus function on vertices of G or T / clusters in Σ , and the total genus of G , T or Σ (Definitions 3.8, 3.23, 3.38) |
| Λ | homology lattice of G , T or Σ (Definitions 3.16, 3.31, 3.48) |
| Λ^\vee | $= \text{Hom}(\Lambda, \mathbb{Z})$, the dual lattice |
| δ | distance function on G , T or Σ (Definitions 3.15, 3.30, 3.45) |
| \tilde{G}, \tilde{T} | core of an open hyperelliptic graph/BY tree (Definitions 3.12, 3.25) |
| T_b, T_y | blue/yellow part of T |
| ι | hyperelliptic involution on a hyperelliptic graph G |
| s | continuous section $G/\langle \iota \rangle \rightarrow G$ to the quotient map (§4.3) |
| $\mathbb{Z}[X]$ | free abelian group on X |

2. BACKGROUND ON METRIC GRAPHS

2.1. Graphs. In what follows, the word *graph* refers to a topological space G homeomorphic to a finite (combinatorial) graph. It comes with a set $V(G)$ of *vertices* (containing all points $x \in G$ of degree $\neq 2$) and *edges* $E(G)$. *Graph isomorphisms* are homotopy classes of homeomorphisms that preserve vertices and edges. Loops and multiple edges are allowed, though note that in the topological setting the action of $\text{Aut } G$ might permute multiple edges and reverse direction of loops.

By a *metric graph* we mean a topological graph G along with a function $l : E(G) \rightarrow \mathbb{R}_{>0}$ which assigns a length to each edge. This may be extended into a metric on G . We write $\delta(v, v')$ for the shortest distance between $v, v' \in V(G)$. We require isomorphisms and automorphisms of metric graphs to preserve lengths.

2.2. Homology of graphs. For a topological space X , $H_i(X)$ denotes the i -th singular homology group of the space X with coefficients in \mathbb{Z} . For a subspace $A \subseteq X$, $H_i(X, A)$ denotes the i -th relative (singular) homology group, again with coefficients in \mathbb{Z} . Most calculations will be carried out via simplicial homology. This will give the same answer where used: see [11, Theorem 2.27]. We refer to Section 2 of op. cit. for more details and proofs of everything outlined below.

Let G be a graph. We make G into a Δ -complex by taking the 0-simplices (resp. 1-simplices) to be the set of vertices (resp. edges, along with their endpoint(s)) of G and write $C_0(G)$ (resp. $C_1(G)$) for the free \mathbb{Z} -module on the 0-simplices (resp. 1-simplices) of G . For each choice of orientation on the edges of G , which for a non-loop edge e amounts to a choice of ‘nose’ e_+ and ‘tail’ e_- , we have an associated boundary map $d : C_1(G) \rightarrow C_0(G)$ sending a non-loop edge e to $e_+ - e_-$, and sending loops to 0. We then have

$$H_1(G) = \text{Ker}(d) \subseteq C_1(G).$$

Given two choices of orientation, there is a canonical isomorphism between the associated homology groups and so $H_1(G)$ is independent of the choice of orientation. For the rest of this section, fix an orientation on G .

2.2.1. Action of automorphisms. Let $\text{Aut } G$ be the group of automorphisms of G . Then $\text{Aut } G$ acts on $C_0(G)$ via its action on the set of vertices, and on $C_1(G)$ via its action on the set of edges save that now we add signs to this action to take account of the orientation. Explicitly, if $\sigma \in \text{Aut } G$ maps the (unsigned) non-loop edge e to e' then we define the action on $e \in C_1(G)$ as

$$\sigma(e) = \begin{cases} e' & \sigma(e_+) = e'_+ \\ -e' & \sigma(e_+) = e'_-. \end{cases}$$

The action on loops is defined similarly, introducing a minus sign if σ maps a loop with its positive orientation to a loop with its negative orientation. This action commutes with the boundary map and induces an action of $\text{Aut } G$ on $H_1(G)$. Whilst the action on $C_1(G)$ depends on the choice of orientation, the induced action on $H_1(G)$ does not (upon canonically identifying the homology groups arising from different choices of orientation).

2.2.2. Length pairing on homology of metric graphs. Suppose now that G is a metric graph with associated length function l . Define the *length pairing* on $C_1(G)$ by setting

$$\langle e, e' \rangle = \begin{cases} l(e) & e = e', \\ 0 & e \neq e', \end{cases}$$

and extending bilinearly. This is independent of the orientation on the edges of G . The induced pairing on $H_1(G)$ is positive definite (since the pairing on $C_1(G)$ is) and invariant under the action of $\text{Aut } G$ (since the same is true of the pairing on $C_1(G)$, for any choice of orientation).

In particular, $H_1(G)$ is a finitely generated, torsion free \mathbb{Z} -module which carries a canonical action of $\text{Aut } G$ and, in the metric case, a positive definite real valued, invariant pairing.

2.2.3. Relative homology. Let H be a subgraph of G (that is, a closed subspace of G which is a union of edges and vertices of G). In the metric case, we put the induced metric on H so that it is also a metric graph. Then H has a natural structure of a Δ -complex inherited from that on G . Having fixed an orientation on the edges of G , we have an induced orientation on

the edges of H . The boundary operator $d : C_1(G) \rightarrow C_0(G)$ then maps $C_1(H)$ into $C_0(H)$ and we have

$$H_1(G, H) = \ker \left(\frac{C_1(G)}{C_1(H)} \longrightarrow \frac{C_0(G)}{C_0(H)} \right).$$

Writing $\text{Aut}_H G$ for the subgroup of automorphisms of G preserving H , the action of $\text{Aut } G$ on $C_1(G)$ defined above induces an action of $\text{Aut}_H G$ on $H_1(G, H)$. Again, this is independent of the choice of orientation on G .

In the metric case, define the *relative length pairing* on $C_1(G)$ by setting

$$\langle e, e' \rangle = \begin{cases} l(e) & e = e', e \notin H, \\ 0 & \text{otherwise,} \end{cases}$$

and extending bilinearly. This induces a positive definite pairing on $H_1(G, H)$ which is invariant under the action of $\text{Aut}_H G$. In particular, as with $H_1(G)$, $H_1(G, H)$ is a finitely generated, torsion free (as $C_1(H)$ is a direct summand of $C_1(G)$) \mathbb{Z} -module equipped with an action of $\text{Aut}_H G$ and, in the metric case, a positive definite real valued, invariant pairing.

For an example illustrating all the definitions above, see Example 3.32.

2.2.4. Subdivision of edges. It will often be convenient to define a new Δ -complex structure on G by ‘subdividing’ certain edges. That is, we take points x_1, \dots, x_m lying on edges of G and define a Δ -complex structure on G by taking the set of 0-simplices to be the set $V(G) \cup \{x_1, \dots, x_m\}$, and redefining the set of 1-simplices accordingly. Let us temporarily denote G , along with the new Δ -complex structure as G^* , and fix an orientation on the 1-simplices of G^* (not necessarily induced from the orientation on the 1-simplices of G). Then there is a natural map $C_1(G) \rightarrow C_1(G^*)$ sending an edge $e \in C_1(G)$ to the (signed) sum over its subdivisions. This map induces an isomorphism $H_1(G) \xrightarrow{\sim} H_1(G^*)$. If the set $\{x_1, \dots, x_m\}$ is preserved by an automorphism σ of G , then σ induces an action on $H_1(G^*)$ by the same formula as in the case of G and the isomorphism $H_1(G) \cong H_1(G^*)$ defined above preserves this action. Moreover, in the metric case, each 1-simplex of G^* has an associated length and we thus obtain a length pairing on $H_1(G^*)$ by the same formula as for G . Since the length of an edge of G is the sum of the lengths of its subdivisions, the isomorphism above identifies the pairings also. Consequently, we will frequently subdivide edges without further comment, with the caveat that when computing actions of automorphisms, we use subdivisions which are preserved by the automorphisms of interest.

The above discussion applies equally well to the relative homology group of G with respect to a subgraph H (along with its automorphism action and, in the metric case, pairing), provided that when defining the simplicial complex structure on G by subdividing edges, we give H the simplicial complex structure inherited from the new one on G as before.

| Type name | G | T | Σ |
|------------------|---------|-----|----------|
| | Genus 0 | | |
| 0 | | | |
| | Genus 1 | | |
| 1 | | | |
| $U_{n,m}$ | | | |
| | Genus 2 | | |
| 2 | | | |
| 1_n | | | |
| $I_{n,m}$ | | | |
| $U_{n,m,k}$ | | | |
| $I_n \times I_m$ | | | |
| $1 \times I_n$ | | | |
| 1×1 | | | |

TABLE 3.1. All type names, hyperelliptic graphs, BY trees and balanced cluster pictures in genus 0, 1 and 2

3. HYPERELLIPTIC GRAPHS, BY TREES AND CLUSTER PICTURES

3.1. Hyperelliptic graphs.

Definition 3.2 (Hyperelliptic graph). Let G be a connected graph equipped with

- $g : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, a function that assigns a *genus* to every vertex,
- $\iota : G \rightarrow G$ an involution (graph isomorphism of order ≤ 2).

We say G (or, more precisely, (G, g, ι)) is a *hyperelliptic graph* if

- (1) vertices of non-zero genus are ι -invariant;
- (2) genus 0 vertices have degree ≥ 3 ;
- (3) $2g(v) + 2 \neq \#$ ι -invariant edges at v , for every vertex v ;
- (4) the topological quotient $G/\langle \iota \rangle$ is contractible (that is, a tree).

In addition, it is convenient to declare two exceptions

[illegible]

(with ι =reflection in the x -axis and vertices of genus 0) to be hyperelliptic graphs as well, although they violate (2).

Definition 3.4 (Open version). An *open* hyperelliptic graph G is a connected graph G with at least one open⁶ edge, a genus function $g : V(G) \rightarrow$

⁶that is, open on one end; such graphs are called ‘graphs with legs’ in [7]

$\mathbb{Z}_{\geq 0}$ and an involution ι , satisfying conditions (1)-(3) of Definition 3.2 for all vertices, and such that the quotient $G/\langle\iota\rangle$ is a tree with a unique open edge (in particular, G has either a unique open edge, or a pair of open edges swapped by ι). In addition, we declare the graph

(3.5) 

(with vertex of genus 0 and ι =reflection in the x -axis) to be an open hyperelliptic graph, though it violates (2).

Remark 3.6. To stress the distinction from the open case, we will sometimes also refer to hyperelliptic graphs (as in Definition 3.2) as *closed hyperelliptic graphs*.

Remark 3.7. Any open hyperelliptic graph is homeomorphic to a connected component of $\bar{G} \setminus I$ for some hyperelliptic graph $\bar{G} = (\bar{G}, g, \iota)$ for $I \subset V(\bar{G})$ an ι -orbit of vertices, together with the induced genus function and action of ι . In other words, G is a hyperelliptic graph with an extra ‘open edge/pair of open edges’. The missing vertex/pair of vertices are referred to as ∞ or ∞^+, ∞^- .

We draw hyperelliptic graphs as follows, with numbers indicating the genus $g(v)$ when it is positive (and omitted when $g(v) = 0$).



Definition 3.8 (Genus). For both closed and open hyperelliptic graphs, the *genus of G* is given by

$$g(G) = \text{rk } H_1(G) + \sum_{v \in V(G)} g(v).$$

Definition 3.9 (Isomorphism). Two hyperelliptic graphs (closed or open) are *isomorphic* if there is a homeomorphism between them that preserves the defining data (vertices, edges, genus markings, commutes with ι). We write $\text{Aut } G$ for the group of *automorphisms* of G . (Recall that they are considered up to homotopy; see §2.1.)

Remark 3.10. In every hyperelliptic graph G that is not the exceptional circle graph from (3.3), ι is the unique involution in $\text{Aut } G$ that fixes all the vertices of positive genus, and such that the quotient $G/\langle\iota\rangle$ is a tree (see e.g. [12] Prop 1.4). In particular, it is central, and graph isomorphisms commute with ι automatically.

Example 3.11. Table 3.1 (2nd column) lists all hyperelliptic graphs of genus 0, 1 and 2, and Table 4.1 open ones of genus 0 and 1, up to isomorphism.

Definition 3.12 (Core). A *(closed) hyperelliptic subgraph H* of a (open or closed) hyperelliptic graph G is a (closed) hyperelliptic graph H such that

- As a topological space, H is a union of vertices and edges of G , closed in G , and closed under ι .
- The vertices of H are exactly those vertices of G that are in H , except for those that have genus 0 and degree 2 in H . The latter become points on the edges of H . (When H is the second exceptional graph (3.3), we declare its two ι -invariant points to be genus 0 vertices.)
- The genus of a vertex of H is the same as its genus in G .

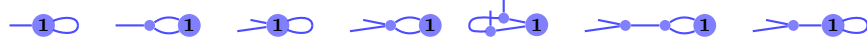
The *core* \tilde{G} of an open hyperelliptic graph G is its maximal closed hyperelliptic subgraph. By Proposition 5.7 below, it is unique, has the same genus as G , and can be easily obtained from G by removing a few vertices and edges near ∞ .

Definition 3.13 (Equivalence). We say that two open hyperelliptic graphs are *equivalent* if they have isomorphic cores.

Example 3.14. Take a single vertex of genus 1 with a loop, and let ι reverse the direction of the loop:

$$\tilde{G} = \text{loop with } \mathbf{1}$$

It is a (closed) hyperelliptic graph \tilde{G} of genus $1+1=2$. There are, up to isomorphism, 7 open hyperelliptic graphs G with \tilde{G} as their core:



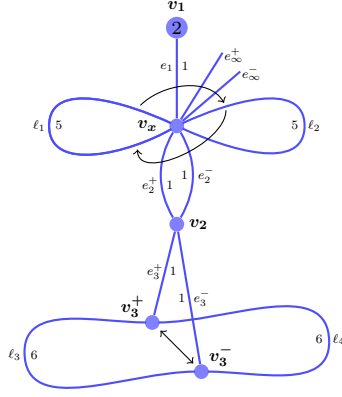
Graphs #1, #2, #5 have $\text{Aut } G = \langle 1, \iota \rangle \cong C_2$, and the other four $\text{Aut } G \cong C_2^2$.

Definition 3.15 (Metric version). A *metric hyperelliptic graph* G (closed or open) is a hyperelliptic graph equipped with an ι -invariant length function $\delta : E(G) \rightarrow \mathbb{R}_{>0}$ on the edges (excluding the open edge). In this case, we write $\delta(v, v')$ for the shortest distance between $v, v' \in V(G)$. We require isomorphisms and automorphisms of metric graphs to preserve δ , and for a hyperelliptic subgraph $H \subset G$ we require⁷ $\delta_H(v, v') = \delta_G(v, v')$ for the vertices of H . Similarly, we say that two open metric hyperelliptic graphs are equivalent if there is an isomorphism between their cores which preserves distance.

Definition 3.16 (The homology lattice Λ). Let G be a closed or open hyperelliptic graph. We set $\Lambda_G = H_1(G)$. Recall from §2.2 that a (closed) hyperelliptic graph comes with a natural action of $\text{Aut } G$ and, if G is a metric hyperelliptic graph, a non-degenerate, real valued, $\text{Aut } G$ -invariant pairing. The same is also true of the open case. Indeed, in Lemma 6.4 we will show that if G is an open hyperelliptic graph with core \tilde{G} , then $H_1(G)$ is canonically isomorphic to $H_1(\tilde{G})$. Since automorphisms of G induce automorphisms of the core via restriction, we may use the closed case to equip Λ_G with an action of $\text{Aut } G$ and, in the metric case, with a non-degenerate, real valued, $\text{Aut } G$ -invariant pairing.

⁷When H is the exceptional circle graph from (3.3), the vertices might change, and so we require instead that the total length of the circle is the same in H and in G .

Example 3.17. Consider the following open hyperelliptic graph G :



The graph has vertices $v_1, v_x, v_2, v_3^+, v_3^-$ of genera $2, 0, 0, 0, 0$; loops ℓ_1, ℓ_2 of lengths $5, 5$; edges $\ell_3, \ell_4, e_1, e_2^+, e_2^-, e_3^+, e_3^-$ of lengths $6, 6, 1, 1, 1, 1, 1, 1$ and two open edges e_∞^+, e_∞^- going off to infinity from v_x . The involution ι fixes v_1, v_2, v_x, e_1 , swaps v_3^+ with v_3^- , e_2^+ with e_2^- , e_3^+ with e_3^- and e_∞^+ with e_∞^- , and reverses the directions of $\ell_1, \ell_2, \ell_3, \ell_4$. G admits an automorphism $\sigma \in \text{Aut}_G$ of order 4, that fixes all vertices except v_3^+ and v_3^- , fixes the edges e_1, e_2^+, e_2^- pointwise, swaps e_3^+ and e_3^- , reverses the directions of ℓ_3 and ℓ_4 , and has an order four action on the loops, sending $\ell_1 \rightarrow -\ell_2 \rightarrow -\ell_1 \rightarrow \ell_2$ (where $-\ell$ denotes ℓ with the opposite orientation).

The lattice Λ_G is

$$\Lambda_G = H_1(G) = \langle \ell_1, \ell_2, \ell_3 + e_3^+ - e_3^-, \ell_4 + e_3^+ - e_3^-, e_2^+ - e_2^- \rangle \simeq \mathbb{Z}^5,$$

where we have picked an orientation for each edge and loop (edges e_2^\pm, e_3^\pm going bottom-to-top, ℓ_3, ℓ_4 right-to-left, and the loops ℓ_1, ℓ_2 oriented clockwise) and where $+$ means concatenation and $-x$ is x with the opposite orientation. The loops in the above basis have lengths $5, 5, 8, 8, 2$, and have trivial intersections, except for the third and fourth basis elements whose intersection has length 2. Thus the length-pairing (see Section 2.2.2) and the action of σ on Λ_G are

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3.2. BY trees.

Definition 3.18 (BY tree). A BY tree is a finite tree T with a genus function $g : V(T) \rightarrow \mathbb{Z}_{\geq 0}$ on vertices and a 2-colouring blue/yellow on vertices and edges such that

- (1) yellow vertices have genus 0, degree ≥ 3 , and only yellow edges;
- (2) blue vertices of genus 0 have at least one yellow edge;
- (3) at every vertex, $2g(v) + 2 \geq \#$ blue edges at v .

Note that all leaves are blue.

In diagrams, yellow edges are drawn squiggly and yellow vertices hollow for the benefit of viewing them in black and white.

Example 3.19. Table 3.1 (third column) lists all BY trees of genus 0, 1 and 2 up to isomorphism.

Notation 3.20. As a topological space, $T = T_b \coprod T_y$ with T_b the *blue part*, and T_y the *yellow part*. (Thus $T_b \subset T$ is a closed subset.)

Definition 3.21 (Open version). An *open BY tree* T is a finite tree T with a unique open edge, a genus function $g : V(T) \rightarrow \mathbb{Z}_{\geq 0}$ on vertices and a 2-colouring blue/yellow on vertices and edges, satisfying conditions (1), (2) and (3) of Definition 3.18.

In other words, as for hyperelliptic graphs, an open BY tree T is a BY tree with one ‘missing’ vertex, that we will refer to as ∞ . Again, we sometimes refer to BY trees of Definition 3.18 as *closed*, to distinguish them from the open ones.

Definition 3.22 (Isomorphic BY trees). Two (closed or open) BY trees are *isomorphic* if there is a homeomorphism between them that preserves the defining data (vertices, edges, genus markings, colouring).

Definition 3.23. For a closed or open BY tree T , the *genus* of T is

$$g(T) = \text{rk } H_1(T, T_b) + \sum_{v \in V(T)} g(v),$$

where the first term is the (singular) relative homology group (see §2.2.3).

Remark 3.24. The relative homology sequence

$$0 = H_1(T) \longrightarrow H_1(T, T_b) \longrightarrow H_0(T_b) \longrightarrow H_0(T) = \mathbb{Z}$$

gives an isomorphism $H_1(T, T_b) \cong \tilde{H}_0(T_b)$ where the latter group is the reduced homology of T_b in degree zero. In particular, $\text{rk } H_1(T, T_b)$ is equal to one less than the number of connected components of T_b .

Definition 3.25 (Core). A *(closed) BY subtree* T' of a (closed or open) BY tree T is a (closed) BY tree T' such that

- As a topological space, T' is a union of vertices and edges of T , and is closed in T .
- The vertices of T' are exactly those vertices of T that are in T' , except for those of genus 0 that in T' have degree 2 and incident edges of the same colour as the vertex. These exceptional vertices become points on the edges of T' .
- The genus of a vertex of T' is the same as its genus on T .

The *core* \tilde{T} of an open BY tree T is its maximal closed BY subtree. Again, we will see later (Proposition 5.7) that it is unique, has the same genus, and is obtained from T by removing a few vertices and edges near ∞ .

Definition 3.26 (Equivalence). We say that two open BY trees are *equivalent* if they have isomorphic cores.

Definition 3.27. An *isomorphism of (closed or open) BY trees* $T \rightarrow T'$ is a pair (α, ϵ) where

- α is a graph isomorphism $T \rightarrow T'$ (in the open case, $T \cup \{\infty\} \rightarrow T' \cup \{\infty\}$ with $\alpha(\infty) = \infty$) that preserves the genera of the vertices and the colours, and
- $\epsilon(Z) = \pm 1$ is a collection of signs for every connected component Z of the yellow part $T_y \subset T$.

Equivalently, ϵ is a collection of signs $\epsilon(v) \in \{\pm 1\}$ and $\epsilon(e) \in \{\pm 1\}$ for every yellow vertex and yellow edge, such that $\epsilon(v) = \epsilon(e)$ whenever e ends at v . The isomorphisms are composed by a cocycle rule

$$(\alpha, \epsilon_\alpha) \circ (\beta, \epsilon_\beta) = (\alpha \circ \beta, \bullet \mapsto \epsilon_\beta(\bullet) \epsilon_\alpha(\beta(\bullet))).$$

An *automorphism* of T is an isomorphism from T to itself. We write $\text{Aut } T$ for the group of automorphisms. (As all ϵ may be chosen to be $+1$, this extended notion of an isomorphism does not affect the earlier definition of BY trees being isomorphic.)

Example 3.28. Take a tree \tilde{T} on 2 blue vertices, one of genus 0, and one of genus 1, with one yellow edge between them:

$$\tilde{T} = \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---}$$

It is a (closed) BY tree of genus $1+1=2$. There are, up to isomorphism, 7 open BY trees T with \tilde{T} as their core:

$$\text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---} \quad \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---} \quad \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---} \quad \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---} \quad \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---} \quad \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---} \quad \text{---} \bullet \text{---} \text{---} \text{---} \bullet \text{---}$$

Trees #1, #2, #5 have $\text{Aut } T = \text{Aut } \tilde{T} \cong C_2$, and the other four $\text{Aut } T \cong C_2^2$.

Remark 3.29. If $T_y^{(1)}, \dots, T_y^{(r)}$ are the connected components of $T_y \subset T$, then, by definition,

$$\text{Aut } T = \text{Aut}_0(T) \ltimes (\mathbb{Z}/2\mathbb{Z})^r,$$

where $\text{Aut}_0(T)$ consists of those elements for which $\epsilon(T_y^{(j)}) = +1$ for all j . Equivalently, $\text{Aut}_0(T)$ is the group of (homotopy classes of) homeomorphisms $T \rightarrow T$ that preserve $V(T), E(T), T_b, T_y$ and g .

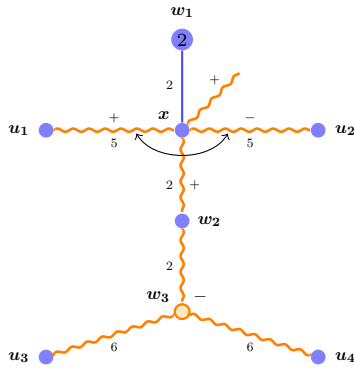
Definition 3.30 (Metric version). A *metric (open or not) BY tree* is one with a length function $\delta : E(T) \rightarrow \mathbb{R}_{>0}$ on the edges (excluding the open one). We denote by $\delta(v, v')$ the distance between $v, v' \in V(T)$, and we require isomorphisms/automorphisms of metric trees to preserve δ . Similarly, we say that two open metric BY trees are equivalent if there is an isomorphism between their cores which preserves distance.

Definition 3.31 (The lattice Λ). Let T be a (open or not) BY tree. We set $\Lambda_T = H_1(T, T_b)$. In the closed case, as detailed in Section 2.2, it comes with a natural action of $\text{Aut}_0 T$ and, if T is a metric BY tree, a non-degenerate, real valued, $\text{Aut}_0 T$ -invariant pairing. We extend the action of $\text{Aut}_0 T$ to an action of the full automorphism group $\text{Aut } T$ as follows. Let $\sigma = (\sigma_0, \epsilon_\sigma)$ be an element of $\text{Aut } T$ and let e be a yellow edge, viewed as an element of $C_1(T)$. Then we set $\sigma(e) = \epsilon(Z)\sigma_0(e)$ where Z is the connected component

of T_y containing e and the action of σ_0 on $C_1(T)$ is as in Section 2.2. This induces the sought action of $\text{Aut } T$ on $H_1(T, T_b)$.

We will show in Lemma 6.4 that if T is an open BY tree with core \tilde{T} , then $H_1(T, T_b)$ is canonically isomorphic to $H_1(\tilde{T}, \tilde{T}_b)$. Since automorphisms of T induce automorphisms of the core via restriction, we may use the closed case to equip Λ_T with an action of $\text{Aut } T$ and, in the metric version, a non-degenerate, real valued, $\text{Aut } T$ -invariant pairing also.

Example 3.32. Consider the following open BY tree T :



The graph T has vertices $u_1, \dots, u_4, w_1, w_2, w_3, x$ of genera $0, 0, 0, 0, 2, 0, 0, 0$. There is an edge from x going off to infinity. T admits an automorphism $\sigma = (\alpha, \epsilon) \in \text{Aut}_T$ of order 4, where α swaps u_1 and u_2 , and fixes all the other vertices and the sign function ϵ is given by

$$\begin{aligned} \epsilon(u_1x) &= 1, & \epsilon(u_2x) &= -1, & \epsilon(w_2x) &= 1, \\ \epsilon(w_3) &= \epsilon(w_3w_2) = \epsilon(u_3w_3) = \epsilon(u_4w_3) &= -1, \end{aligned}$$

where yz denotes the edge between the vertices y and z .

The lattice $\Lambda_T = H_1(T, T_b)$ is the relative homology of T with respect to its blue part. In other words it consists of 1-chains in T whose boundary is blue (see Section 2.2.3). Writing $[a, b]$ for the shortest path from a to b , and

$$c_1 = [u_1, x], \quad c_2 = [u_2, x], \quad c_3 = [u_3, w_2], \quad c_4 = [u_4, w_2], \quad c_5 = [w_2, x],$$

the lattice is given by

$$\Lambda_T = H_1(T, T_b) = \langle c_1, c_2, c_3, c_4, c_5 \rangle \simeq \mathbb{Z}^5.$$

The cycles have lengths 5, 5, 8, 8, 2, and have trivial intersections, except for the third and fourth basis elements whose intersection has length 2. Clearly α swaps c_1 and c_2 and fixes the other basis vectors, while the sign ϵ is 1 on c_1 and c_5 , and -1 on c_2, c_3, c_4 . Thus the length-pairing (see Section 2.2.2) and the action of σ on Λ_T are

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3.3. Cluster pictures.

Definition 3.33 (Cluster picture). Let X be a finite set and $\Sigma \subset \mathcal{P}(X)$ a collection of non-empty subsets of X ; elements of Σ are called *clusters*. Then Σ (or (X, Σ)) is a *cluster picture* if

- (1) Every singleton ('root') is a cluster, and X is a cluster.

(2) Two clusters are either disjoint or contained in one another.

We say that Σ has *genus* g if $|X| \in \{2g+1, 2g+2\}$.

Two cluster pictures (X, Σ_X) and (Y, Σ_Y) are *isomorphic* if there is a bijection $X \rightarrow Y$ that takes Σ_X to Σ_Y . (So we may take $X = \{1, \dots, n\}$, and just consider its cluster pictures, up to S_n -permutations.)

Remark 3.34. Let $X = \{r_1, \dots, r_n\} \subset K$ be a finite subset of a field with a (non-trivial) valuation. Then the non-empty subsets of X that are cut out by discs in \bar{K} form a cluster picture. Conversely, every cluster picture arises in this way: for any K that has at least n elements in its residue field one can find $X \subset K$ that realises it.

Example 3.35. Let $X = \{1, 2, 3, 4, 5, 6\}$ and let

$$\Sigma = \left\{ \{1\}, \dots, \{6\}, \{5, 6\}, X \right\}.$$

Thus, apart from the required singletons and X , there is one extra cluster $\mathfrak{s} = \{5, 6\}$. We draw Σ with ovals around every $\mathfrak{s} \in \Sigma$ with $|\mathfrak{s}| > 1$:

$$\Sigma = \text{ (diagram showing 6 singletons and one cluster of 2 elements, all enclosed in ovals) }$$

It is a cluster picture of genus 2. In the language of the last remark, it is realised, for example, by $\{1, 2, 3, 4, -p, p\} \subset \mathbb{Q}_p$ ($p \geq 5$).

Definition 3.36 (Children). If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster, we write $\mathfrak{s}' < \mathfrak{s}$ and refer to \mathfrak{s}' as a *child* of \mathfrak{s} , and \mathfrak{s} as the *parent* of \mathfrak{s}' .

Definition 3.37 (Types of clusters). A cluster \mathfrak{s} is *proper* if $|\mathfrak{s}| > 1$ or $|\mathfrak{s}| = |X| = 1$, a *twin* if $|\mathfrak{s}| = 2$, and it is *odd/even* if its size is odd/even. A proper cluster is *übereven* if it has no odd children.

Definition 3.38 (Genus). A non-übereven cluster has *genus* $g = g(\mathfrak{s})$ if it has $2g+1$ or $2g+2$ odd children; übereven clusters are declared to have genus 0.

Definition 3.39 (Balanced). A cluster picture Σ is *balanced* if $|X|$ is even, X is the only cluster of size $> \frac{|X|}{2}$, and there are either 0 or 2 clusters of size $\frac{|X|}{2}$.

Example 3.40. Table 3.1 (last column) lists all balanced cluster pictures of genus 0, 1 and 2 up to isomorphism.

Definition 3.41. An *isomorphism* $\Sigma \rightarrow \Sigma'$ is an equivalence class of pairs (α, ϵ) with $\alpha : \Sigma \rightarrow \Sigma'$ a bijection that preserves cluster sizes and inclusions, and $\epsilon(\mathfrak{s}) = \pm 1$ a collection of signs for even clusters $\mathfrak{s} \in \Sigma$, such that

$$\epsilon(\mathfrak{s}') = \epsilon(\mathfrak{s}) \quad \text{for } \mathfrak{s} \text{ übereven, } \mathfrak{s}' < \mathfrak{s}.$$

Here we say pairs (α, ϵ) and (α', ϵ') are equivalent if α and α' induce the same map between the sets of proper clusters of Σ and Σ' , and $\epsilon = \epsilon'$. We compose isomorphisms by a cocycle rule

$$(\alpha, \epsilon_\alpha) \circ (\beta, \epsilon_\beta) = (\alpha \circ \beta, \mathfrak{s} \mapsto \epsilon_\beta(\mathfrak{s})\epsilon_\alpha(\beta(\mathfrak{s}))).$$

An *automorphism* of Σ is an isomorphism from Σ to itself, and we write $\text{Aut } \Sigma$ for the group of automorphisms. (As all ϵ may be chosen to be $+1$, this extended notion of an isomorphism does not affect the definition of being isomorphic.) Equivalently, one may think of an element of $\text{Aut } \Sigma$ as a pair $(\sigma, \epsilon_\sigma)$ where σ is a permutation of the proper clusters of Σ and ϵ_σ is a collection of signs as above, composition of two such being given by the same formula. We will frequently take this viewpoint without further comment in what follows.

Remark 3.42. As for BY trees (Remark 3.29), we have

$$\text{Aut } \Sigma = \text{Aut}_0(\Sigma) \ltimes (\mathbb{Z}/2\mathbb{Z})^r,$$

where $\text{Aut}_0(\Sigma)$ consists of elements for which $\epsilon(\mathfrak{s}) = +1$ for all even clusters $\mathfrak{s} \in \Sigma$. The number r is the number of equivalence classes of even clusters for the equivalence relation generated by ' $\mathfrak{s}' \sim \mathfrak{s}$ if \mathfrak{s} is *übereven* and $\mathfrak{s}' < \mathfrak{s}$ '.

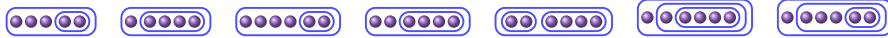
Definition 3.43 (Equivalence). We say that cluster pictures (X_1, Σ_1) and (X_2, Σ_2) are *equivalent* if (X_2, Σ_2) is isomorphic to a cluster picture obtained from (X_1, Σ_1) in a finite number of the following steps $(X, \Sigma) \rightarrow (X', \Sigma')$.

- (i) ('add cocluster') $X' = X$, $\Sigma' = \Sigma \cup \{X \setminus \mathfrak{s}\}$ for some $\mathfrak{s} < X$, $|X| = 2g + 2$.
- (ii) ('remove cocluster') $X' = X$, $\Sigma' = \Sigma \setminus \{\mathfrak{s}\}$ for some $\mathfrak{s} < X$ with $|\mathfrak{s}| \geq 2$, when X has exactly two children, $|X| = 2g + 2$.
- (iii) (' $2g+1 \rightarrow 2g+2$ ') $X' = X \coprod \{r\}$, $\Sigma' = \Sigma \cup \{X'\} \cup \{\{r\}\}$, when $|X| = 2g + 1$.
- (iv) (' $2g+2 \rightarrow 2g+1$ ') $X = X' \coprod \{r\}$, $\Sigma = \Sigma' \cup \{X\} \cup \{\{r\}\}$, when $|X| = 2g + 2$.

In general, note that

- the genus of a cluster picture is preserved under equivalence;
- (ii)=(i) $^{-1}$ and (iv)=(iii) $^{-1}$, hence this is an equivalence relation;
- (i) does nothing when X has only 2 children;

Example 3.44. There are, up to isomorphism, 7 cluster pictures $\tilde{\Sigma}$ in the equivalence class of Σ from Example 3.35:



Cluster pictures #1, #2, #5 have $\text{Aut } \Sigma \cong C_2$, and the other four $\text{Aut } \Sigma \cong C_2^2$. All 7 have trivial $\text{Aut}_0(\Sigma)$, so the automorphisms come from choices of signs on even clusters.

As for hyperelliptic graphs and BY trees, we have a metric version:

Definition 3.45 (Metric version). A cluster picture (X, Σ) is *metric* when every pair of proper clusters $\mathfrak{s} < \mathfrak{r}$ is assigned a *distance* $\delta(\mathfrak{s}, \mathfrak{r}) = \delta(\mathfrak{r}, \mathfrak{s}) \in \mathbb{R}_{>0}$. The distance function clearly extends to every pair of proper clusters: if \mathfrak{s} and \mathfrak{r} are distinct proper clusters with least common ancestor \mathfrak{u} (no child of \mathfrak{u} contains both \mathfrak{s} and \mathfrak{r} , but \mathfrak{u} does), so that

$$\mathfrak{s} < \mathfrak{s}_1 < \dots < \mathfrak{s}_{k-1} < \mathfrak{u} > \mathfrak{r}_{m-1} > \dots > \mathfrak{r}_1 > \mathfrak{r},$$

we let $\delta(\mathfrak{s}, \mathfrak{r})$ be the sum of the $k + m$ distances between adjacent clusters in the chain; we let $\delta(\mathfrak{s}, \mathfrak{s}) = 0$.

An *isomorphism as metric cluster pictures* is one that preserves δ .

We say that two metric cluster pictures are *equivalent* if one is (up to isomorphism) obtained from the other by a finite number of ‘metric versions’ of the moves (i)-(iv) of Definition 3.43, in which we allow any metric δ' on (X', Σ') such that

- $\delta'(\mathfrak{r}, \mathfrak{r}') = \delta(\mathfrak{r}, \mathfrak{r}')$ for every $\mathfrak{r}, \mathfrak{r}' \in \Sigma \cap \Sigma'$ with (for moves (i)-(ii)) $\mathfrak{r}, \mathfrak{r}' \neq X$,
- $\delta'(\mathfrak{r}, X \setminus \mathfrak{s}) = \delta(\mathfrak{r}, X)$ for all $\mathfrak{r} \in \Sigma' \setminus \{X, X \setminus \mathfrak{s}\}$ in move (i),
- $\delta'(\mathfrak{r}, X) = \delta(\mathfrak{r}, \mathfrak{s})$ for all $\mathfrak{r} \in \Sigma' \setminus \{X\}$ in move (ii).

In diagrams, distances are shown using subscripts on clusters: a cluster gets a subscript indicating the distance to its parent (cf. Example 3.49).

Remark 3.46. The clusters arising in Remark 3.34 are naturally metric cluster pictures. In this setting, for a cluster \mathfrak{s} we define its ‘depth’ as $\delta(\mathfrak{s}) = \min_{r, r' \in \mathfrak{s}} (\text{val}_K(r - r'))$, and for a pair $\mathfrak{s} < \mathfrak{s}'$ set the distance to be given by the ‘relative depth’, $\delta(\mathfrak{s}, \mathfrak{s}') = \delta(\mathfrak{s}) - \delta(\mathfrak{s}')$. This is the same as the formula given in §1.3 in terms of the diameter.

We now define the lattice Λ_Σ for a cluster picture (X, Σ) . We need one preliminary piece of notation.

Notation 3.47. Let (X, Σ) be a cluster picture. Let

$$\mathcal{E}_\Sigma = \{\text{even, non-übereven clusters } \mathfrak{s} \neq X\}.$$

Further, for $\mathfrak{s} \in \mathcal{E}_\Sigma$, write $\hat{\mathfrak{s}}$ for the smallest non-übereven cluster strictly containing \mathfrak{s} . If no such cluster exists, we set $\hat{\mathfrak{s}} = X$.

Definition 3.48 (The lattice Λ). Let (X, Σ) be a cluster picture. If X is not an übereven cluster, let

$$\Lambda_\Sigma = \mathbb{Z}[\mathcal{E}_\Sigma] = \left\{ \sum_{\mathfrak{s} \in \mathcal{E}_\Sigma} \lambda_{\mathfrak{s}} \mathfrak{s} \mid \lambda_{\mathfrak{s}} \in \mathbb{Z} \right\}.$$

If X is übereven, let

$$\Lambda_\Sigma = \left\{ \sum_{\mathfrak{s} \in \mathcal{E}_\Sigma} \lambda_{\mathfrak{s}} \mathfrak{s} \in \mathbb{Z}[\mathcal{E}_\Sigma] \mid \sum_{\hat{\mathfrak{s}}=X} \lambda_{\mathfrak{s}} = 0 \right\}.$$

Further, define an action of $\sigma = (\sigma_0, \epsilon_\sigma) \in \text{Aut } \Sigma$ on Λ_Σ by

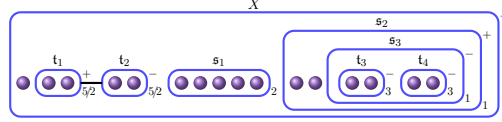
$$\sigma \cdot \mathfrak{s} = \epsilon_\sigma(\mathfrak{s}) \sigma_0(\mathfrak{s}).$$

In the metric case, define a pairing on Λ_Σ by setting, for $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathcal{E}_\Sigma$,

$$\langle \mathfrak{s}_1, \mathfrak{s}_2 \rangle = \begin{cases} 2\delta(\mathfrak{s}_1 \wedge \mathfrak{s}_2, \hat{\mathfrak{s}}_1) & \hat{\mathfrak{s}}_1 = \hat{\mathfrak{s}}_2, \\ 0 & \hat{\mathfrak{s}}_1 \neq \hat{\mathfrak{s}}_2, \end{cases}$$

where $\mathfrak{s}_1 \wedge \mathfrak{s}_2$ denotes the least common ancestor of \mathfrak{s}_1 and \mathfrak{s}_2 .

Example 3.49. Consider the following cluster picture Σ :



The cluster picture Σ has clusters t_1, \dots, t_4 (twins), s_1, s_2, s_3 , X of genera $0, 0, 0, 0, 2, 0, 0, 0$ with distances between children and parents $\delta(t_1, X) = \delta(t_2, X) = 5/2$, $\delta(t_3, s_3) = \delta(t_4, s_3) = 3$, $\delta(s_3, s_2) = 1$, $\delta(s_2, X) = 1$, $\delta(s_1, X) = 2$.

This cluster picture admits an automorphism $\sigma = (\alpha, \epsilon) \in \text{Aut}_\Sigma$ of order 4, where σ swaps t_1 and t_2 (which is indicated by the black line between the two leftmost twins) and fixes all other clusters. The sign function ϵ is

$$\begin{aligned} \epsilon(t_1) = 1, \quad \epsilon(t_2) = -1, \quad \epsilon(s_2) = 1, \quad \epsilon(X) = 1, \\ \epsilon(t_3) = \epsilon(t_4) = \epsilon(s_3) = -1, \end{aligned}$$

indicated by the $+$ and $-$ on top of the respective even clusters.

Here the set of even non-übereven clusters (that are not X) is $\mathcal{E}_\Sigma = \{t_1, t_2, t_3, t_4, s_2\}$, and since X is not übereven,

$$\Lambda_\Sigma = \langle t_1, t_2, t_3, t_4, s_2 \rangle \simeq \mathbb{Z}^5.$$

By definition

$$\hat{t}_1 = \hat{t}_2 = \hat{s}_2 = X \quad \text{and} \quad \hat{t}_3 = \hat{t}_4 = \hat{s}_2,$$

and

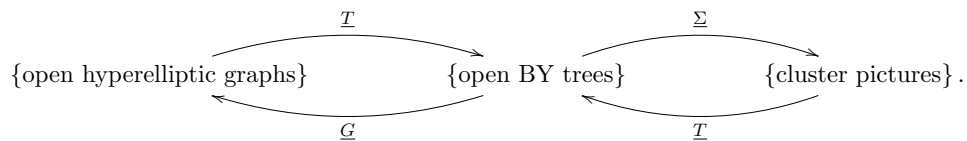
$$t_1 \wedge t_2 = t_1 \wedge s_2 = t_2 \wedge s_2 = X \quad \text{and} \quad t_3 \wedge t_4 = s_3.$$

Thus by definition of the pairing on Λ_Σ and since the action of σ on Λ_Σ is a signed permutation given by α and ϵ ,

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 2 & 0 \\ 0 & 0 & 2 & 8 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. ONE-TO-ONE CORRESPONDENCE (OPEN CASE)

In this section we explain how to pass between cluster pictures, hyperelliptic graphs and BY trees. We construct maps \underline{G} , \underline{T} and $\underline{\Sigma}$ (which become inverse to each other upon passing to isomorphism classes) between objects according to the following diagram:



| G | T | Σ | Aut | Λ |
|--------------------------|-----|----------|------------------|--------------|
| Core 0 (genus 0) | | | | |
| | | | C_1 | 0 |
| | | | C_2 | 0 |
| Core 1 (genus 1) | | | | |
| | | | C_1 | 0 |
| | | | C_2 | 0 |
| | | | C_2 | 0 |
| Core $U_{n,m}$ (genus 1) | | | | |
| | | | C_2 | \mathbb{Z} |
| | | | $C_2 \times C_2$ | \mathbb{Z} |
| | | | $C_2 \times C_2$ | \mathbb{Z} |
| | | | $C_2 \times C_2$ | \mathbb{Z} |

TABLE 4.1. Open hyperelliptic graphs, open BY trees and cluster pictures in genus 0 and 1 up to isomorphism

In addition, for \underline{F} one of \underline{G} , \underline{T} or $\underline{\Sigma}$, and X an object on which it is defined, we construct a map $\underline{F} : \text{Aut } X \rightarrow \text{Aut } \underline{F}(X)$, which turns out to be an isomorphism in each case. Here there is a subtlety: for an open hyperelliptic graph G , the map $\text{Aut } G \rightarrow \text{Aut } \underline{T}(G)$ depends on a choice of section $G/\langle \iota \rangle \rightarrow G$. The exact dependence of the map on the choice is examined in Proposition 4.7. By contrast, all other maps on automorphism groups are canonical. The constructions are summarised in Table 4.20 and illustrated in Examples 4.21 and 4.22.

The main result is the following.

Theorem 4.2. *The maps \underline{G} , \underline{T} and $\underline{\Sigma}$ defined in Constructions 4.4, 4.8, 4.13 and 4.15 give a genus preserving one-to-one correspondence between isomorphism classes of (either metric or not)*

- (i) *Cluster pictures,*
- (ii) *Open hyperelliptic graphs,*
- (iii) *Open BY trees.*

Moreover, the associated maps on automorphism groups (see Constructions 4.4, 4.8, 4.13 and 4.15 and the preceding paragraph) are isomorphisms.

Proof. Combine Proposition 4.11 (‘hyperelliptic graphs \leftrightarrow BY trees’) with Proposition 4.19 (‘BY trees \leftrightarrow cluster pictures’). \square

4.1. Hyperelliptic graphs \leftrightarrow BY trees. We begin with the maps between open hyperelliptic graphs and open BY trees, as well as the associated

maps on automorphism groups. In fact, the constructions apply equally well in the closed version. Since both versions will be relevant later, we cover both here. Before detailing the constructions, we briefly discuss the notion of a ‘section to the quotient map’ for a hyperelliptic graph.

Remark 4.3 (Sections to the quotient map). Let G be a (open or not) hyperelliptic graph and $\pi : G \rightarrow G/\langle\iota\rangle$ be the quotient map. In order to construct the map between automorphisms of G and automorphisms of the associated BY tree $T = G/\langle\iota\rangle$, it will be necessary to choose a continuous map $s : T \rightarrow G$ such that $\pi \circ s = \text{id}$. That is, a continuous section to π (we henceforth refer to s as a *section*, the continuity being understood). More concretely, such a choice amounts to the following. Write $G_y = \pi^{-1}(T_y)$ and choose a decomposition $G_y = G_y^+ \amalg G_y^-$, such that $\pi : G_y^\pm \rightarrow T_y$ is a homeomorphism (that is, a choice of ‘top’ and ‘bottom’ above every connected component of T_y). Such a choice determines a section s by sending $x \in T_b$ to its unique preimage in G , and $x \in T_y$ to its unique preimage in G_y^+ . Conversely, a section s determines such a decomposition by taking G_y^+ to be $s(T_y)$.

Construction 4.4 ($\underline{T}(G)$).

Objects: Let (G, g, ι) be a (closed) hyperelliptic graph, without loss of generality considered to be metric. Define

$$T = \underline{T}(G) = G/\langle\iota\rangle,$$

the topological quotient. It is a tree by Definition 3.2 (4), and we colour the branch locus T_b of the quotient map $\pi : G \rightarrow T$ blue and $T_y = T \setminus T_b$ yellow. In other words, under π , blue points have one preimage and yellow points have two preimages. We make T into a graph as follows:

Write $V(G) = \{v_1, \dots, v_k, v_{k+1}, \iota(v_{k+1}), \dots, v_n, \iota(v_n)\}$, with the first k vertices ι -invariant, followed by the pairs swapped by ι . Then v_1, \dots, v_k give vertices $\bar{v}_1, \dots, \bar{v}_k \in V(T)$, each \bar{v}_j declared blue of genus $g(v_j)$, and each pair $\{v_j, \iota(v_j)\}$ for $j > k$ gives one vertex $\bar{v}_j \in V(T)$, declared yellow of genus 0.

Next, take an edge of G , say of length d . It is either mapped to another edge by ι , is ι -invariant, or is ι -anti-invariant. Then

- each ι -invariant edge vw gives a blue edge $\bar{v}\bar{w}$ of length $2d$;
- each ι -anti-invariant edge $v\iota(v)$ (allowing for $\iota(v) = v$ in the case of loops) gives a yellow edge of length d from \bar{v} to an extra blue genus 0 leaf of T ;
- each swapped pair of edges vw and $\iota(v)\iota(w)$ gives a yellow edge $\bar{v}\bar{w}$ of length $2d$.

Finally, if G is an open (possibly metric) hyperelliptic graph, we define $\underline{T}(G)$ in the same way (treating w as ∞ for the open edge(s)).

Automorphisms: Let G , T and $\pi : G \rightarrow T$ be as above. To define the map $\text{Aut } G \rightarrow \text{Aut } T$ we begin by choosing a section $s : T \rightarrow G$ to π . Now given

$\sigma \in \text{Aut } G$ we construct an automorphism $\underline{T}(\sigma) = (\underline{T}(\sigma)_0, \epsilon_{\underline{T}(\sigma)})$ of T as follows (if we wish to record the choice of section, we write $\underline{T}(\sigma; s)$). Since σ necessarily commutes with the hyperelliptic involution ι (see Remark 3.10), it induces a graph-theoretic automorphism $\bar{\sigma}$ of T which preserves genus and colour by construction. We set $\underline{T}(\sigma)_0 = \bar{\sigma}$. Next, let Z be a connected component of T_y . Then we define

$$\epsilon_{T(\sigma; s)}(Z) = \begin{cases} 1 & s(\bar{\sigma}(Z)) = \sigma(s(Z)), \\ -1 & \text{else.} \end{cases}$$

Proposition 4.5. *If G is a hyperelliptic graph, then $\underline{T}(G)$ is a BY tree. The same holds in the open case.*

Proof. It is easy to check the claim in genus 0 and 1 (see Tables 3.1, 4.1), so assume $g > 1$ from now on. We check the three conditions of Definition 3.18:

(1) *Yellow vertices have genus 0, degree ≥ 3 , and only yellow edges:* all yellow vertices come from pairs $v_j, \iota(v_j)$ consisting of two distinct vertices of G swapped by ι , and are declared to have genus 0 in the construction. Moreover, yellow vertices have only yellow outgoing edges, because blue edges only come from edges between ι -invariant vertices. Finally, $G \rightarrow T$ is a two-to-one topological cover at \bar{v}_j , and so $\deg \bar{v}_j = \deg v_j \geq 3$ by Definition 3.2 (2).

(2) *Blue vertices of genus 0 have at least one yellow edge:* blue vertices either come from loops, in which case they have a yellow edge by construction, or from ι -invariant vertices v_j . Suppose $g(v_j) = 0$ but \bar{v}_j has only blue outgoing edges. Then $\deg v_j \geq 3$ by 3.2 (2), so v_j has at least three ι -invariant outgoing edges, contradicting Definition 3.2 (3).

(3) *At every vertex, $2g(v) + 2 \geq \#$ blue edges at v :* if v is yellow, or a blue leaf coming from a loop, then it has only yellow edges and there is nothing to prove. If v is blue, the blue edges from \bar{v} are in one-to-one correspondence with the ι -invariant edges from v , and the claim follows from Definition 3.2 (3). \square

We now examine the dependence of the map $T(-, s) : \text{Aut } G \rightarrow \text{Aut } T$ constructed above on the choice of section s . For this it will be useful to have the following definition.

Definition 4.6. Let G be a hyperelliptic graph (open or not), let $T = \underline{T}(G)$ be the associated BY tree, and let $\pi : G \rightarrow T$ be the quotient map. Further, let s and s' be two sections to π . We define the automorphism $\psi_{s, s'} = (\text{id}, \epsilon_{s, s'})$ of T by setting, for a component Z of T_y ,

$$\epsilon_{s, s'}(Z) = \begin{cases} 1 & s(Z) = s'(Z) \\ -1 & s(Z) \neq s'(Z). \end{cases}$$

Proposition 4.7. *Let G be a hyperelliptic graph (possibly open), let $T = \underline{T}(G)$ be the associated BY tree, and let $\pi : G \rightarrow T$ be the quotient map. Then*

for each section $s : T \rightarrow G$ to π , the map $\underline{T}(-; s)$ defines a homomorphism $\text{Aut } G \rightarrow \text{Aut } T$. Moreover, if s and s' are two choices of section then for all $\sigma \in \text{Aut } G$ we have

$$\underline{T}(\sigma; s') = \psi_{s,s'} \circ \underline{T}(\sigma; s) \circ \psi_{s,s'}^{-1}.$$

In particular, the map $\underline{T}(-; s)$ depends on s only up to conjugation.

Proof. It is immediate from the construction that for any automorphism σ of G , $\underline{T}(\sigma; s)$ is indeed an automorphism of T and it is clear that $\underline{T}(-; s)$ is a homomorphism. A straightforward calculation verifies the statement concerning the dependence on the choice of section. \square

Construction 4.8 ($\underline{G}(T)$).

Objects: Let T be a BY tree, viewed as a topological space. Let G be the topological space given by glueing two disjoint copies T^+ and T^- of T along their common blue parts. Then G comes with a natural map $\pi : G \rightarrow T$ making it into a two-to-one cover of T ramified along T_b , as well as an involution ι (swapping the elements of the fibres of π over points in T_y) such that $G/\langle \iota \rangle = T$. When T has genus ≥ 2 , we make G into a graph as follows:

- A blue vertex $\bar{v} \in V(T)$ which is not a genus 0 leaf gives an ι -invariant vertex $v \in V(G)$ of genus $g(\bar{v})$.
- a yellow vertex $\bar{v} \in V(T)$ gives vertices v^+ and v^- in $V(G)$ (where $v^+ \in T^+$ and $v^- \in T^-$) swapped by ι , of genus 0,
- a (necessarily blue) genus 0 leaf \bar{v} of T with an edge from \bar{v} to a blue vertex \bar{w} , of length d , gives an ι -anti-invariant loop on w of length d ,
- a (blue) genus 0 leaf \bar{v} of T joined by an edge of length d to a yellow vertex \bar{w} gives an ι -anti-invariant edge between w^+ and w^- of length d ,
- a blue edge between (necessarily blue vertices) \bar{v} and \bar{w} of length d gives an ι -invariant edge between v and w of length $d/2$,
- a yellow edge between \bar{v} and \bar{w} of length d gives a pair of edges, one between v^+ and w^+ and one between v^- and w^- , swapped by ι , each of length $d/2$ (here, if \bar{v} (resp. \bar{w}) is blue we set $v^+ = v^- = v$ (resp. $w^+ = w^- = w$)).

When T is an open BY tree, we use exactly the same construction (say, by adding a vertex at ∞ , following the steps above, and removing the vertices above ∞).

Finally, when T has genus 0 or 1, we have to declare vertices of G slightly differently, and we refer to Tables 3.1, 4.1 for the correspondence.

Automorphisms: Write G_b for those points in G fixed by ι and G_y for $G \setminus G_b$. Further, write G_y^+ for the points in G_y which come from T^+ and G_y^- for the points coming from T^- . To ease notation, for $x \in G$ we also set $\bar{x} := \pi(x) \in T$.

Now let $\sigma = (\sigma_0, \epsilon_\sigma)$ be an automorphism of T . We define the automorphism $\underline{G}(\sigma)$ of G as follows:

- for $x \in G_b$ we set $\underline{G}(\sigma)(x)$ to be the unique point of G lying over $\sigma(\bar{x})$
- for $x \in G_y^+$ we set

$$\underline{G}(\sigma)(x) = \begin{cases} \pi^{-1}(\{\bar{x}\}) \cap G_y^+ & \epsilon(\bar{x}) = 1 \\ \pi^{-1}(\{\bar{x}\}) \cap G_y^- & \epsilon(\bar{x}) = -1. \end{cases}$$

- similarly, for $x \in G_y^-$ we set

$$\underline{G}(\sigma)(x) = \begin{cases} \pi^{-1}(\{\bar{x}\}) \cap G_y^- & \epsilon(\bar{x}) = 1 \\ \pi^{-1}(\{\bar{x}\}) \cap G_y^+ & \epsilon(\bar{x}) = -1. \end{cases}$$

(In the above, for $\bar{x} \in T$ yellow, we write $\epsilon_\sigma(\bar{x})$ for the value of ϵ_σ on the connected component of T_y containing \bar{x} .)

Note that σ commutes with the hyperelliptic involution ι .

Proposition 4.9. *If T is a BY tree, then $\underline{G}(T)$ is a hyperelliptic graph. The same holds in the open case. Further, the map $\underline{T}(-)$ gives a homomorphism $\text{Aut } T \rightarrow \text{Aut } G$.*

Proof. That $\underline{G}(T)$ is a hyperelliptic graph follows by reversing the argument of Proposition 4.5. The claim about automorphisms is clear. \square

Remark 4.10. As constructed, for a BY tree T , $G = \underline{G}(T)$ comes equipped with a canonical section $s : T \rightarrow G$ sending $\bar{x} \in T$ to x^+ (in the notation of Construction 4.8). In general, suppose that $s' : T \rightarrow G$ is any section. Then given $\sigma = (\sigma_0, \epsilon_\sigma) \in \text{Aut } T$ we may define an automorphism $\underline{G}(\sigma; s') \in \text{Aut } G$ by first defining it on the image of s' as

$$\underline{G}(\sigma; s')(s'(\bar{x})) = \begin{cases} s'(\sigma_0(\bar{x})) & \bar{x} \text{ blue or } \bar{x} \text{ yellow and } \epsilon_\sigma(\bar{x}) = 1, \\ \iota(s'(\sigma_0(\bar{x}))) & \text{else,} \end{cases}$$

and then extending to the whole of G by insisting that $\underline{G}(\sigma; s')$ commutes with the hyperelliptic involution. Then $\underline{G}(\sigma; s)$ agrees with $\underline{G}(\sigma)$ as defined in Construction 4.8. Moreover, writing $\phi_{s,s'}$ for the automorphism of G such that $s' = \phi_{s,s'} \circ s$, we have

$$\underline{G}(\sigma; s') = \phi_{s,s'} \circ \underline{G}(\sigma; s) \circ \phi_{s,s'}^{-1}$$

for all $\sigma \in \text{Aut } T$.

The following proposition establishes the correspondence between hyperelliptic graphs and BY trees.

Proposition 4.11. *Let G be a hyperelliptic graph, possibly open and/or metric and let $T = \underline{T}(G)$ be the corresponding BY tree. Then*

- (1) *We have an equality of genera $g(G) = g(T)$,*
- (2) *For any choice of section s , the map $\underline{T}(-; s)$ gives an isomorphism $\text{Aut } G \xrightarrow{\sim} \text{Aut } T$.*
- (3) *We have $\underline{G}(T) \cong G$ (as metric graphs if G is metric).*

Conversely, let T be a BY tree, possibly open and/or metric, and let $G = \underline{G}(T)$ be the corresponding hyperelliptic graph. Then

- (1)' We have an equality of genera $g(T) = g(G)$,
- (2)' The map $\underline{G}(-)$ gives an isomorphism $\text{Aut } T \xrightarrow{\sim} \text{Aut } G$,
- (3)' We have $\underline{T}(G) \cong T$ (as metric BY trees if T is metric).

Proof. First let G be a hyperelliptic graph and T the associated BY tree. For concreteness we consider the closed non-metric case, the argument being identical in the other cases. It is clear from the constructions that we have $\underline{G}(T) \cong G$ (non-canonically). We will show later in Proposition 6.6 that there is an isomorphism $H_1(G) \cong H_1(T, T_b)$. Since $\pi : G \rightarrow T$ is also a bijection on the vertices of positive genus, we have $g(T) = g(G)$ (cf. Definitions 3.8, 3.23). We have now established (1) and (3).

To show (2), since changing the section s serves to conjugate $\underline{T}(-; s)$, it suffices to prove the result for a single section s , which we now fix. If $\sigma \in \text{Aut } G$ is such that $\underline{T}(\sigma; s)$ is the trivial automorphism of T , then σ acts trivially on the quotient $G/\langle \iota \rangle$ and also preserves the section s . Such an automorphism is easily seen to be the identity so $\underline{T}(-; s)$ is injective. To show surjectivity it now suffices to show that $|\text{Aut } T| \leq |\text{Aut } G|$. By (3), it suffices to show that $\underline{G}(-) : \text{Aut } T \rightarrow \text{Aut } \underline{G}(T)$ is injective which we do independently below. Modulo this remaining claim, this completes the proof of (1), (2) and (3).

Now let T be a BY tree (say closed and non-metric) and $G = \underline{G}(T)$. Again, it is clear from the construction that we have $\underline{T}(G) \cong T$ (and now the isomorphism is canonical) so (3)' is proven. Part (1)' now follows from (3)' and (1).

We now show (2)'. Let $\sigma = (\sigma_0, \epsilon_\sigma)$ be an automorphism of T . If $\underline{G}(\sigma)$ is trivial then it is immediate from the definition that $\underline{G}(\sigma)$ is an automorphism of G that commutes with the hyperelliptic involution and acts trivially on $s(T)$. Since such automorphisms are necessarily trivial, $\underline{G}(\sigma)$ is injective (which also completes the proof of (2)). As above, the isomorphism of (3)' along with the injectivity of $\underline{T}(-) : G \rightarrow \underline{T}(G)$ shown previously forces $\underline{G}(\sigma)$ to be an isomorphism. \square

Remark 4.12. It is tempting to define ‘signed hyperelliptic graphs’ as pairs (G, s) where G is a hyperelliptic graph (possibly open) and $s : G/\langle \iota \rangle \rightarrow G$ is a section to the quotient map, with isomorphisms between two such pairs (G, s) and (G', s') required to take s onto s' . It is easy to see that two such pairs (G, s) and (G', s') are isomorphic if and only if G and G' are, and that \underline{G} and \underline{T} give a one-to-one correspondence between signed hyperelliptic graphs and BY trees in which there are no choices involved in identifying automorphism groups (automorphisms of signed hyperelliptic graphs which forget the section must be allowed though). Additionally, the isomorphism $G \cong \underline{G}(\underline{T}(G))$ becomes canonical. We have decided against setting up the correspondence this way, however, as for our application to hyperelliptic

curves, the hyperelliptic graphs we obtain do not come with a natural section s and so it is convenient to allow an arbitrary choice.

4.2. Cluster pictures \leftrightarrow BY trees. We now construct the maps between cluster pictures and open BY trees.

Construction 4.13 ($\underline{T}(\Sigma)$).

Objects: Let (X, Σ) be a cluster picture. Define $T = \underline{T}(\Sigma)$ to be the open BY tree whose vertices are

- one vertex $v_{\mathfrak{s}}$ for every proper cluster \mathfrak{s} which is not a twin, coloured yellow if \mathfrak{s} is übereven and blue otherwise.
- one blue vertex (a leaf) $v_{\mathfrak{t}}$ for every twin \mathfrak{t} .

For the edges

- for every pair $\mathfrak{s}' < \mathfrak{s}$ (see Definition 3.36) with \mathfrak{s}' proper, link $v_{\mathfrak{s}'}$ and $v_{\mathfrak{s}}$ by an edge, yellow if \mathfrak{s}' is even and blue otherwise.
- add one open edge $v_X \infty$, yellow if X is even and blue otherwise.

In the metric version, set the length to be $\delta(\mathfrak{s}, \mathfrak{s}')$ for blue edges and $2\delta(\mathfrak{s}, \mathfrak{s}')$ for yellow edges.

Finally, define the genus of a vertex $v_{\mathfrak{s}}$ to be the genus of the cluster \mathfrak{s} as in Definition 3.38.

Automorphisms: Let $\sigma = (\sigma_0, \epsilon_\sigma)$ be an element of $\text{Aut } \Sigma$, where σ_0 is viewed as a permutation of the proper clusters. Then we define an automorphism $\underline{T}(\sigma) = (\underline{T}(\sigma)_0, \epsilon_{\underline{T}(\sigma)})$ of $T = \underline{T}(\Sigma)$ as follows. For a vertex $v_{\mathfrak{s}}$ of T , set $\underline{T}(\sigma)_0(v_{\mathfrak{s}}) = v_{\sigma(\mathfrak{s})}$. To define $\epsilon_{\underline{T}(\sigma)}$, for a yellow component T_y of T , pick a yellow edge from $v_{\mathfrak{s}}$ to $v_{\mathfrak{s}'}$ in T_y where we take $v_{\mathfrak{s}'}$ to be nearer to ∞ (since yellow vertices have only yellow edges, each component of T_y has at least one yellow edge). Then \mathfrak{s} is an even cluster and we set $\epsilon_{\underline{T}(\sigma)}(T_y) = \epsilon_\sigma(\mathfrak{s})$. The compatibility of signs on even clusters as in Definition 3.33 ensures that this is well defined.

Proposition 4.14. *Let (X, Σ) be a cluster picture. Then*

- (1) $\underline{T}(\Sigma)$ is an open BY tree,
- (2) The map $\underline{T}(-)$ defines a homomorphism $\text{Aut } \Sigma \rightarrow \text{Aut } \underline{T}(\Sigma)$.

Proof. (1) Let us check the conditions of a BY tree (Definition 3.21). When $|X| \leq 2$, this is easy to check by hand. Otherwise:

Yellow vertices have genus 0, degree ≥ 3 , and only yellow edges: übereven clusters have genus 0 and at least 2 children, giving at least 2 edges and an edge to the parent or to ∞ . So there are at least 3 outgoing edges, all of which are yellow since all children of an übereven cluster are even.

Blue vertices of genus 0 have at least one yellow edge: every even cluster has a yellow edge to its parent or ∞ , and every odd cluster of size > 2 has either positive genus or at least one even child that gives a yellow edge.

At every vertex, $2g(v) + 2 \geq \#$ blue edges at v : every vertex v comes either from a twin or from a cluster \mathfrak{s} of size > 2 . In the former case, v has no blue edges. In the latter,

- if \mathfrak{s} is odd, it has an odd number ($=2g(\mathfrak{s}) + 1$) of odd children, and

$$2g(\mathfrak{s}) + 2 = 1 + \#\{\text{odd children}\} \geq 1 + \#\{\text{odd proper children}\}$$

$$= \# \text{ blue edges at } v.$$
- if \mathfrak{s} is even, it has an even number ($=2g(\mathfrak{s}) + 2$) of odd children, and

$$2g(\mathfrak{s}) + 2 = \#\{\text{odd children}\} \geq \#\{\text{odd proper children}\}$$

$$= \# \text{ blue edges at } v.$$

(2) If $\sigma = (\sigma_0, \epsilon_\sigma)$ is an automorphism of Σ then σ_0 preserves inclusion and the size of proper clusters by definition. Thus $\underline{T}(\sigma)$ preserves adjacency, colour and genus and it is now clear that $\underline{T}(\sigma)$ is an automorphism of $\underline{T}(\Sigma)$. That the map $\underline{T}(-)$ is an homomorphism follows from the way we have defined composition of automorphisms for cluster pictures and BY trees (cf. Definitions 3.27, 3.41). \square

The construction in the opposite direction is as follows.

Construction 4.15 ($\underline{\Sigma}(T)$).

Objects: Let T be an open (possibly metric) BY tree. Define a partial order on the vertices of T by setting $v \preceq v'$ if v' lies on the unique shortest path from v to ∞ .

For each blue vertex $v \in T_b$, let $\deg_{T_b}(v)$ be the number of blue edges at v (i.e. the degree of v in T_b) and set $m_v = 2g(v) + 2 - \deg_{T_b}(v)$, which is non-negative by Definition 3.21 (3). Take m_v singletons $x_{v,1}, \dots, x_{v,m_v}$ and define $X_v := \{x_{v,1}, \dots, x_{v,m_v}\}$. Now take $X = \bigcup_{v \in T_b} X_v$.

Further, for every vertex v of T (of any colour), set

$$\mathfrak{s}_v = \bigcup_{v' \preceq v, v' \text{ blue}} X_{v'}$$

and define the subset $\Sigma \subseteq \mathcal{P}(X)$ as

$$\Sigma = \bigcup_{v \in T} \{\mathfrak{s}_v\} \cup \bigcup_{x \in X} \{x\}.$$

Set $\underline{\Sigma}(T) = (X, \Sigma)$, the cluster picture associated to T .

In the metric case, for $e \in E(T)$, write

$$l(e) = \begin{cases} \delta(e) & e \text{ blue,} \\ \frac{1}{2}\delta(e) & e \text{ yellow,} \end{cases}$$

and extend to a distance function of T in the obvious way. Now for vertices $v, w \in T$, define $\delta(\mathfrak{s}_v, \mathfrak{s}_w) = l(v, w)$.

Automorphisms: Let $\sigma = (\sigma_0, \epsilon_\sigma) \in \text{Aut}(T)$. Then we define an element $\underline{\Sigma}(\sigma) = (\underline{\Sigma}(\sigma)_0, \epsilon_{\underline{\Sigma}(\sigma)})$ of $\text{Aut } \Sigma$ as follows. Noting that the map $v \mapsto \mathfrak{s}_v$ is a bijection between the vertices of T and the proper clusters of Σ , define a permutation $\underline{\Sigma}(\sigma)_0$ of the proper clusters of Σ by setting $\underline{\Sigma}(\sigma)_0(\mathfrak{s}_v) = \mathfrak{s}_{\sigma_0(v)}$. Since $\mathfrak{s}_v \subseteq \mathfrak{s}_{v'}$ if and only if $v \preceq v'$, this preserves inclusion. To define $\epsilon_{\underline{\Sigma}(\sigma)}$, let $\mathfrak{s} \in \Sigma$ be an even cluster. Then $\mathfrak{s} = \mathfrak{s}_v$ for a vertex v of T and we'll see in Corollary 4.18 (2) below that the edge from v towards ∞ is yellow.

Writing Z for the connected component of T_y containing this edge, we define $\epsilon_{\underline{\Sigma}(\sigma)}(\mathfrak{s}) = \epsilon_\sigma(Z)$.

Proposition 4.16. *Let T be an open BY tree. Then $\underline{\Sigma}(T)$ is a cluster picture. Moreover, $\underline{\Sigma}(-)$ defines a homomorphism $\text{Aut } T \rightarrow \text{Aut } \underline{\Sigma}(T)$.*

Proof. In both cases this is clear from Construction 4.15. \square

Given a vertex v of an open BY tree T , it is not obvious from Construction 4.15 how the size of the associated cluster \mathfrak{s}_v relates to invariants of T and v . The following two results explain this, as well as showing that Construction 4.15 preserves genus.

Proposition 4.17. *Let T be an open BY tree, e its unique open edge, and $\underline{\Sigma}(T) = (X, \Sigma)$ the associated cluster picture. Then*

$$|X| = \begin{cases} 2g(T) + 2 & \text{if } e \text{ is yellow,} \\ 2g(T) + 1 & \text{if } e \text{ is blue.} \end{cases}$$

In particular, $g(T) = g(\Sigma)$, and $|X|$ is even if and only if e is yellow.

Proof. By Construction 4.15 we see that

$$|X| = \sum_{v \in T_b} (2g(v) + 2 - \deg_{T_b}(v)).$$

Now since yellow vertices have genus 0, we may split this sum as

$$|X| = \sum_{v \in T} 2g(v) + \sum_{v \in T_b} (2 - \deg_{T_b}(v)).$$

If e is yellow then T_b is a disjoint union of closed connected trees, hence

$$\sum_{v \in T_b} (2 - \deg_{T_b}(v)) = 2|V(T_b)| - 2|E(T_b)| = 2\#\{\text{connected comps. of } T_b\}.$$

On the other hand, if e is blue, it is counted one fewer times in the sum. We thus obtain

$$|X| = 2 \left(\sum_{v \in T} g(v) + \#\{\text{connected comps. of } T_b\} \right) - \begin{cases} 0 & \text{if } e \text{ is yellow,} \\ 1 & \text{if } e \text{ is blue.} \end{cases}$$

Since $\text{rk} H_1(T, T_b)$ is equal to one less than the number of connected components of T_b (see Remark 3.24) the result follows. \square

Corollary 4.18. *Let T be an open BY tree and $\Sigma = \underline{\Sigma}(T)$ the associated cluster picture. Fix a vertex $v \in T$ and (in the notation Construction 4.15) let $\mathfrak{s}_v \in \Sigma$ be the associated cluster. Further, denote by e_v the edge from v towards ∞ . Then:*

(1) *We have*

$$|\mathfrak{s}_v| = \begin{cases} 2g(T_v) + 2 & \text{if } e_v \text{ is yellow,} \\ 2g(T_v) + 1 & \text{if } e_v \text{ is blue,} \end{cases}$$

where here T_v denotes the open BY tree generated by the vertices $v' \preceq v$ of T along with the open edge e_v ,

- (2) The cluster \mathfrak{s}_v is even if and only if e_v is yellow, and $\ddot{u}b\ddot{e}r\ddot{e}v\ddot{e}n$ if and only if v itself is yellow,
- (3) We have an equality of genera $g(\mathfrak{s}_v) = g(v)$.

Proof. The claims are easy to check when T has genus 0 or 1, so assume the genus is at least 2.

(1). Apply Proposition 4.17 to the open BY tree T_v .

(2). That \mathfrak{s}_v is even if and only if e_v is yellow is clear from (1). Next, since yellow vertices have only yellow edges and no associated singletons, it is clear that if v is yellow then \mathfrak{s}_v is $\ddot{u}b\ddot{e}r\ddot{e}v\ddot{e}n$. For the converse it is convenient to first observe that if v is blue then the number of odd children of \mathfrak{s}_v is either $2g(v) + 2$ or $2g(v) + 1$, the former case occuring if and only if e_v is yellow. Indeed, by Construction 4.15 the number of children of \mathfrak{s}_v of size 1 is given by $m_v = 2g(v) + 2 - \deg_{T_b}(v)$, whilst part (1) applied to the vertices adjacent to v shows that the number of odd proper children of \mathfrak{s}_v is given by the number of blue edges at v , excluding the edge e_v towards infinity (should this be blue).

Since $g(v)$ is non negative, it now follows that if v is blue then \mathfrak{s}_v cannot be $\ddot{u}b\ddot{e}r\ddot{e}v\ddot{e}n$.

(3). If $v \in T$ is yellow then \mathfrak{s}_v is $\ddot{u}b\ddot{e}r\ddot{e}v\ddot{e}n$ by (2) and both $g(v)$ and $g(\mathfrak{s}_v)$ are 0. Suppose now that $v \in T$ is blue. As above, the number of odd children of \mathfrak{s}_v is either $2g(v) + 1$ or $2g(v) + 2$. It is now immediate from the definition of the genus of a cluster that $g(\mathfrak{s}_v) = g(v)$ as claimed. \square

The following Proposition completes the proof of Theorem 4.2.

Proposition 4.19. *Let T be an open (possibly metric) BY tree and $(X, \Sigma) = \underline{\Sigma}(T)$ be the associated cluster picture. Then*

- (1) *we have an equality of genera $g(T) = g(\Sigma)$.*
- (2) *The map $\sigma \mapsto \underline{\Sigma}(\sigma)$ gives an isomorphism $\text{Aut } T \xrightarrow{\sim} \text{Aut } \Sigma$.*
- (3) *We have $\underline{T}(\Sigma) \cong T$ (as metric BY trees if T is metric).*

Conversely, let (X, Σ) be a (possibly metric) cluster picture and let $T = \underline{T}(\Sigma)$ be the associated open BY tree. Then

- (1)' *we have an equality of genera $g(\Sigma) = g(T)$.*
- (2)' *The map $\sigma \mapsto \underline{T}(\sigma)$ gives an isomorphism $\text{Aut } \Sigma \xrightarrow{\sim} \text{Aut } T$.*
- (3)' *We have $\underline{\Sigma}(T) \cong \Sigma$ (as metric cluster pictures if Σ is metric).*

Proof. We consider the non-metric case throughout, the metric case being an easy extension. First let T be an open BY tree and $(X, \Sigma) = \underline{\Sigma}(T)$ the associated cluster picture. Part (1) was shown previously in Proposition 4.17.

Next we show part (3). In the notation of Constructions 4.13 and 4.15, consider the map $f : T \rightarrow \underline{T}(\Sigma)$ sending $v \in T$ to $v_{\mathfrak{s}_v}$. It is clear from the constructions that this is a graph theoretic isomorphism. Moreover, by

Construction 4.13 and Corollary 4.18 (2) we see that f preserves colour (of both edges and vertices). Finally, to see that f preserves genus, fix $v \in T$. Since for $\mathfrak{s} \in \Sigma$ we defined the genus of $v_{\mathfrak{s}}$ to be the genus of the cluster \mathfrak{s} , it suffices to show that $g(v) = g(\mathfrak{s}_v)$ for each $v \in T$, which is Corollary 4.18 (3).

To show (2), let $\sigma = (\sigma_0, \epsilon_\sigma) \in \text{Aut } T$ and suppose that $\underline{\Sigma}(\sigma)$ is trivial in $\text{Aut } \Sigma$. Then as $v \mapsto \mathfrak{s}_v$ is a bijection between the vertices of T and the proper clusters of Σ , σ_0 fixes every vertex of T . Moreover, if $\epsilon_{\underline{\Sigma}(\sigma)}$ is trivial on each even cluster then ϵ_σ must be trivial on each yellow edge and is then itself trivial. This shows that $\underline{\Sigma}(-)$ is injective and, in particular, that we have $|\text{Aut } T| \leq |\text{Aut } \Sigma|$. To show that $\underline{\Sigma}(-)$ is an isomorphism, it suffices to show that we also have the reverse inequality. In light of (3), it suffices to show that the map $\underline{T}(-) : \text{Aut } \Sigma \rightarrow \text{Aut } \underline{T}(\Sigma)$ is injective, which we do below.

We now turn to (1)', (2)' and (3)', for which we fix a cluster picture (X, Σ) and let $T = \underline{T}(\Sigma)$ be the associated BY tree. We first show (3)'. In the notation of Constructions 4.13 and 4.15, we'll show that the map $h : \mathfrak{s} \mapsto \mathfrak{s}_{v_{\mathfrak{s}}}$ is an isomorphism of cluster pictures. It is clear that it gives a bijection on proper clusters which preserves inclusion. To complete the argument, we prove by induction that $|\mathfrak{s}| = |\mathfrak{s}_{v_{\mathfrak{s}}}|$ for all proper clusters \mathfrak{s} .

First suppose that \mathfrak{s} is a minimal proper cluster, i.e. all its children are singletons. Then by the definition of the genus of \mathfrak{s} we have

$$|\mathfrak{s}| = \begin{cases} 2g(\mathfrak{s}) + 1 & \mathfrak{s} \text{ odd,} \\ 2g(\mathfrak{s}) + 2 & \mathfrak{s} \text{ even.} \end{cases}$$

Now $v_{\mathfrak{s}}$ is blue and has no children in T by minimality of \mathfrak{s} . Moreover, its parent edge is yellow if \mathfrak{s} is even and blue if \mathfrak{s} is odd. In particular we have

$$\deg_{T_b}(v_{\mathfrak{s}}) = \begin{cases} 1 & \mathfrak{s} \text{ odd,} \\ 0 & \mathfrak{s} \text{ even,} \end{cases}$$

whence $m_{v_{\mathfrak{s}}} = |\mathfrak{s}_{v_{\mathfrak{s}}}| = |\mathfrak{s}|$ as desired (here, as in Construction 4.15, for any vertex v of T we set $m_v = 2g(v) + 2 - \deg_{T_b}(v)$).

Next, take a proper cluster $\mathfrak{s} \in \Sigma$ and suppose that we have $|\mathfrak{s}'| = |\mathfrak{s}_{v_{\mathfrak{s}'}}|$ for all proper clusters $\mathfrak{s}' \subseteq \mathfrak{s}$. In particular, to show that $|\mathfrak{s}| = |\mathfrak{s}_{v_{\mathfrak{s}}}|$, it suffices to show that \mathfrak{s} and $\mathfrak{s}_{v_{\mathfrak{s}}}$ both have the same number of children of size 1. Now by Corollary 4.18 and the discussion on genera in the proof of (3), it follows that \mathfrak{s} and $\mathfrak{s}_{v_{\mathfrak{s}}}$ have the same genus and parity. Combining the inductive hypothesis with the observation that the genus and parity of a cluster together determine how many odd children it has (cf. Definition 3.38) completes the proof.

Part (1)' now follows upon combining (1) and (3)'.

Finally, we show (2)'. In light of (3)' and the injectivity of the map in (2) shown above, it suffices to show that $\sigma \mapsto \underline{T}(\sigma)$ is injective (and this also completes the proof that the map in (2) is an isomorphism). This follows by

noting that $\mathfrak{s} \mapsto v_{\mathfrak{s}}$ is a bijection between proper clusters of Σ and vertices of T , and that even sized clusters give vertices whose parent edge is yellow, so triviality of $\epsilon_{\underline{T}(\sigma)}$ forces triviality of ϵ_{σ} . \square

TABLE 4.20. Dictionary for the correspondence (open case, genus ≥ 1)

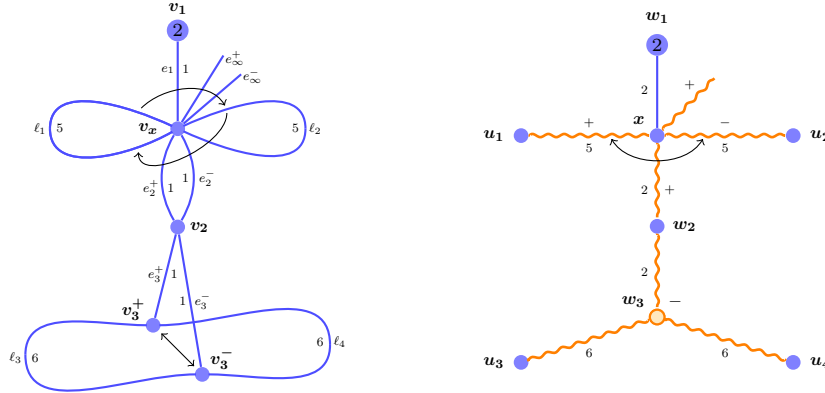
| Cluster picture Σ | Open BY tree T | Open hyperelliptic graph G |
|--|---|--|
| cluster \mathfrak{s} with $ \mathfrak{s} > 2$ of genus g • \mathfrak{s} odd • \mathfrak{s} even non-übereven • \mathfrak{s} übereven | vertex $w_{\mathfrak{s}}$ not a genus 0 leaf of genus g with parent edge $a_{\mathfrak{s}}$ • $w_{\mathfrak{s}}$ blue $a_{\mathfrak{s}}$ blue • $w_{\mathfrak{s}}$ blue $a_{\mathfrak{s}}$ yellow • $w_{\mathfrak{s}}$ yellow $a_{\mathfrak{s}}$ yellow | ι -orbit of vertices $\{v_{\mathfrak{s}}\}$ or $\{v_{\mathfrak{s}}^+, v_{\mathfrak{s}}^-\}$ of genus g with ι -orbit of parent edge(s) $p_{\mathfrak{s}}$ or $\{p_{\mathfrak{s}}^+, p_{\mathfrak{s}}^-\}$ • $v_{\mathfrak{s}}$ with parent edge $p_{\mathfrak{s}}$ • $v_{\mathfrak{s}}$ with parent edges $p_{\mathfrak{s}}^+, p_{\mathfrak{s}}^-$ • $v_{\mathfrak{s}}^+, v_{\mathfrak{s}}^-$ with parent edges $p_{\mathfrak{s}}^+, p_{\mathfrak{s}}^-$ |
| $\mathfrak{s} < \mathfrak{s}'$ $\delta(\mathfrak{s}, \mathfrak{s}') = \begin{cases} 2d & \mathfrak{s} \text{ odd} \\ d & \mathfrak{s} \text{ even} \end{cases}$ | $a_{\mathfrak{s}} = w_{\mathfrak{s}} w_{\mathfrak{s}'}$ length $2d$ | $p_{\mathfrak{s}} = v_{\mathfrak{s}} v_{\mathfrak{s}'}$ or $p_{\mathfrak{s}}^{\pm} = v_{\mathfrak{s}} v_{\mathfrak{s}'}^{\pm}$ or $v_{\mathfrak{s}}^{\pm} v_{\mathfrak{s}'}$ or $v_{\mathfrak{s}} v_{\mathfrak{s}'}^{\pm}$ or $v_{\mathfrak{s}}^{\pm} v_{\mathfrak{s}'}^{\pm}$ length d |
| twin \mathfrak{t} $\mathfrak{t} < \mathfrak{s}$ $\delta(\mathfrak{t}, \mathfrak{s}) = d/2$ | genus 0 leaf $u_{\mathfrak{t}}$ with yellow edge $a_{\mathfrak{t}}$ to $w_{\mathfrak{s}}$ of length d | an edge $\ell_{\mathfrak{t}}$ $\ell_{\mathfrak{t}} = v_{\mathfrak{s}} v_{\mathfrak{s}}$ or $\ell_{\mathfrak{t}} = v_{\mathfrak{s}}^+ v_{\mathfrak{s}}^-$ of length d |
| $(\sigma, \epsilon) \in \text{Aut } \Sigma$ | $(\sigma, \epsilon) \in \text{Aut } T$ | $\sigma \in \text{Aut } G$ |
| $\sigma : \mathfrak{s} \mapsto \mathfrak{s}_2$ $\epsilon(\mathfrak{s}), \quad \text{for } \mathfrak{s} \text{ even}$ | $\sigma : w_{\mathfrak{s}} \mapsto w_{\mathfrak{s}_2}$ $\epsilon(a_{\mathfrak{s}}), \quad \text{for } a_{\mathfrak{s}} \text{ yellow}$ | $\sigma : v_{\mathfrak{s}} \mapsto v_{\mathfrak{s}_2}$ or $\sigma : v_{\mathfrak{s}}^{\pm} \mapsto v_{\mathfrak{s}_2}^{\pm \epsilon(v_{\mathfrak{s}}^*)}$ $\sigma : p_{\mathfrak{s}} \mapsto p_{\mathfrak{s}_2}$ or $\sigma : p_{\mathfrak{s}}^{\pm} \mapsto p_{\mathfrak{s}_2}^{\pm \epsilon(v_{\mathfrak{s}}^*)}$ $\epsilon(v_{\mathfrak{s}}^*) \in \{\pm 1\}$ |
| $\sigma : \mathfrak{t} \mapsto \mathfrak{t}_2, \epsilon(\mathfrak{t})$ | $\sigma : u_{\mathfrak{t}} \mapsto u_{\mathfrak{t}_2}, \epsilon(a_{\mathfrak{t}})$ | $\sigma : \ell_{\mathfrak{t}} \mapsto \epsilon(\ell_{\mathfrak{t}}) \ell_{\mathfrak{t}_2}, \epsilon(\ell_{\mathfrak{t}}) \in \{\pm 1\}$ (where $-\ell_{\mathfrak{t}}$ is $\ell_{\mathfrak{t}}$ with reversed orientation) |

4.3. Summary of constructions and examples. The one-to-one correspondence given by Theorem 4.2 is easy to use in practice. Table 4.20 summarizes the constructions (it follows from Constructions 4.4, 4.8, 4.13 and Proposition 4.19). We illustrate how to use it in Examples 4.21 and 4.22.

As in Remark 4.3, the hyperelliptic graph described in the third column of Table 4.2 comes with the decomposition $G_y = G_y^+ \amalg G_y^-$, where G_y^+ consists of all edges and vertices denoted with a $+$. (Thus in order to construct automorphisms of a cluster picture or a BY tree from that of a hyperelliptic graph, it is first necessary to pick such a decomposition.)

Recall that in a BY tree, from every vertex v there is a shortest path towards ∞ . The *parent edge* of v is the edge a incident to v on this path. Similarly, for a vertex v of a hyperelliptic graph, a *parent edge* is an edge incident to v on one of the shortest paths towards ∞ .

Example 4.21. Consider the open hyperelliptic graph G and the open BY tree T together with the automorphisms σ_G and (α_T, ϵ_T) of Examples 3.17 and 3.32:



Following Table 4.20, from G we form the associated BY tree by creating:

- blue vertices w_1, x, w_2 of genera 2,0,0 corresponding to the ι -invariant vertices v_1, v_x, v_2 of G ,
- a yellow vertex w_3 corresponding to $\{v_3^+, v_3^-\}$,
- blue genus 0 leaves u_1, u_2, u_3, u_4 corresponding to the loops ℓ_1, ℓ_2 and the ι -anti-invariant edges ℓ_3, ℓ_4 ,
- a blue edge from w_1 to x of length 2 corresponding to ι -invariant edge e_1 ,
- yellow edges from w_2 to x and w_3 to w_2 of lengths 2,2 corresponding to the ι -orbits $\{e_2^+, e_2^-\}$ and $\{e_3^+, e_3^-\}$,
- yellow edges from u_1 to x , u_2 to x , u_3 to w_3 , u_4 to w_3 of lengths 5,5,6,6 corresponding to $\ell_1, \ell_2, \ell_3, \ell_4$,
- a yellow open edge from x to ∞ corresponding to the two open edges from v_x to ∞ .

This construction precisely yields T .

To compute the automorphism (α, ϵ) corresponding to σ_G , consider the hyperelliptic graph G with the decomposition of its ι -permuted part $G_y = G_y^+ \amalg G_y^-$, where G_y^+ consists of $e_2^+, e_3^+, v_3^+, e_\infty^+$ and the top halves of ℓ_1, ℓ_2, ℓ_3 and ℓ_4 (call these $\ell_1^+, \ell_2^+, \ell_3^+, \ell_4^+$).

Then (α, ϵ) is given as follows: α acts on T by swapping u_1 and u_2 , since σ_G swaps ℓ_1 and ℓ_2 and fixes all other ι -orbits, and

$$\begin{aligned} \sigma_G(\ell_1^+) = \ell_2^+ &\implies \epsilon(u_1x) = 1, \\ \sigma_G(\ell_2^+) = \ell_1^+ &\implies \epsilon(u_2x) = -1, \\ \sigma_G(e_\infty^+) = e_\infty^+ &\implies \epsilon(x\infty) = 1, \\ \sigma_G(e_2^+) = e_2^+ &\implies \epsilon(xw_2) = 1, \\ \sigma_G(v_3^+) = v_3^- &\implies \epsilon(u_3w_3) = \epsilon(u_4w_3) = \epsilon(w_2w_3) = \epsilon(w_3) = -1. \end{aligned}$$

In other words, we obtain $(\alpha, \epsilon) = (\alpha_T, \epsilon_T)$.

From T , to form the associated hyperelliptic graph, we create:

- vertices v_1, v_x, v_2 fixed by ι of genera 2,0,0, corresponding to the blue vertices w_1, x, w_2 ,
- vertices v_3^+, v_3^- swapped by ι , of genera 0,0 corresponding to the yellow vertex w_3 ,
- two edges e_∞^+, e_∞^- swapped by ι , corresponding the open yellow edge from x to ∞ ,
- an edge from v_1 to v_x of length 1, corresponding to the blue edge from w_1 to x ,
- two edges e_2^+, e_2^- swapped by ι , of lengths 1, corresponding to the yellow edge from the blue vertex w_2 to the blue vertex x ,
- two edges e_3^+, e_3^- swapped by ι , of lengths 1, corresponding to the yellow edge from the yellow vertex w_3 to the blue vertex w_2 ,
- two ι -anti-invariant edges ℓ_1, ℓ_2 from v_x to itself of lengths 5, corresponding to the yellow edges from the blue vertex x to the genus 0 leaves u_1 and u_2 ,
- two ι -anti-invariant edges ℓ_3, ℓ_4 from v_3^+ to v_3^- of lengths 6, corresponding to the yellow edges from the yellow vertex w_3 to the genus 0 leaves u_3 and u_4 .

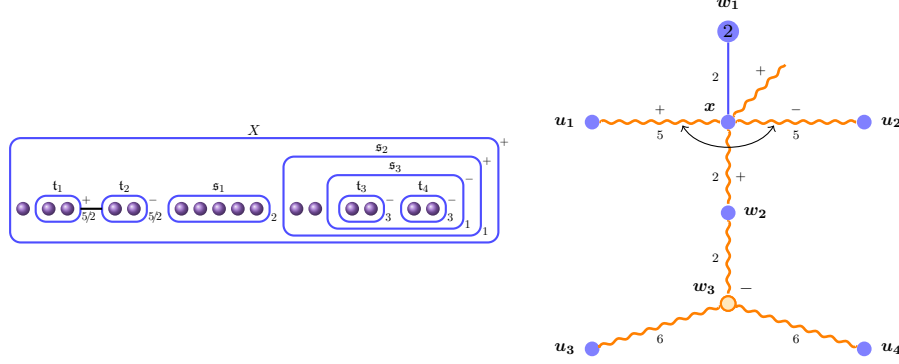
We obtain precisely G .

The automorphism σ corresponding to $(\alpha_T, \epsilon_T) \in \text{Aut}(T)$ is given by

- σ fixes v_1, v_x, v_2 and preserves $\{v_3^+, v_3^-\}$ since α_T fixes w_1, x, w_2 and w_3 ,
- σ fixes $e_\infty^+, e_\infty^-, e_2^+, e_2^-$ since α_T fixes $w_2x, x\infty$ and ϵ_T is 1 on both edges,
- σ swaps e_3^+ and e_3^- , v_3^+ and v_3^- and changes the orientation of ℓ_3 and ℓ_4 since α_T fixes $w_3w_2, w_3, u_3w_3, u_4w_3$ and ϵ_T is -1 on all of them,
- σ maps ℓ_1 to ℓ_2 since α_T maps u_1 to u_2 and $\epsilon_T(u_1x) = 1$,
- σ maps ℓ_2 to $-\ell_1$ since α_T maps u_2 to u_1 and $\epsilon_T(u_2x) = -1$ (where $-\ell$ denotes a loop ℓ with opposite orientation).

We then obtain $\sigma = \sigma_G$.

Example 4.22. Consider the cluster picture Σ and the open BY tree T together with the automorphisms $(\alpha_\Sigma, \epsilon_\Sigma)$ and (α_T, ϵ_T) from Examples 3.49 and 3.32:



Following Table 4.20, from Σ we construct the associated BY tree by creating:

- blue vertices w_1, x, w_2 of genera 2, 0, 0, corresponding to the non-übereven clusters $\mathfrak{s}_1, X, \mathfrak{s}_2$ of size > 2 ,
- a yellow vertex w_3 of genus 0 corresponding to the übereven cluster \mathfrak{s}_3 ,
- blue genus 0 leaves u_1, u_2, u_3, u_4 corresponding to the twins $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4$,
- a yellow open edge from x to ∞ , since the top cluster X is even,
- a blue edge from w_1 to x of length 2, since \mathfrak{s}_1 is odd, $\mathfrak{s}_1 < X$ and $\delta(\mathfrak{s}_1, X) = 2$,
- yellow edges from u_1 to x , u_2 to x , w_2 to x , w_3 to w_2 , u_3 to w_3 and u_4 to w_3 of lengths 5, 5, 2, 2, 6, 6, since $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{t}_3, \mathfrak{t}_4$ are even, $\mathfrak{t}_1 < X$, $\mathfrak{t}_2 < X$, $\mathfrak{s}_2 < X$, $\mathfrak{s}_3 < \mathfrak{s}_2$, $\mathfrak{t}_3 < \mathfrak{s}_3$, $\mathfrak{t}_4 < \mathfrak{s}_3$, and $\delta(\mathfrak{t}_1, X) = \frac{5}{2}$, $\delta(\mathfrak{t}_2, X) = \frac{5}{2}$, $\delta(\mathfrak{s}_3, \mathfrak{s}_2) = 1$, $\delta(\mathfrak{s}_2, X) = 1$, $\delta(\mathfrak{t}_3, \mathfrak{s}_3) = 3$ and $\delta(\mathfrak{t}_4, \mathfrak{s}_3) = 3$.

This yields T . The automorphism (α, ϵ) corresponding to $(\alpha_\Sigma, \epsilon_\Sigma)$ is given by

- α swaps u_1 and u_2 and fixes all other vertices, since α_Σ swaps \mathfrak{t}_1 and \mathfrak{t}_2 and fixes all other clusters,
- ϵ is 1 on the yellow edges $x\infty$, u_1x , w_2x , since $\epsilon_\Sigma(X) = \epsilon_\Sigma(\mathfrak{t}_1) = \epsilon_\Sigma(\mathfrak{s}_2) = 1$,
- ϵ is -1 on the yellow edges u_2x , w_3w_2 , u_3w_3 , u_4w_3 , since $\epsilon_\Sigma(\mathfrak{t}_2) = \epsilon_\Sigma(\mathfrak{s}_3) = \epsilon_\Sigma(\mathfrak{t}_3) = \epsilon_\Sigma(\mathfrak{t}_4) = -1$,
- $\epsilon(w_3) = -1$ since it inherits the sign from its parent edge.

This shows precisely that $(\alpha, \epsilon) = (\alpha_T, \epsilon_T)$.

From T we construct the associated cluster picture by creating:

- clusters $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_x$ of size > 2 and genera 2, 0, 0, 0 corresponding to vertices that are not genus 0 leaves w_1, w_2, w_3, w_x ,
- twins $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4$ corresponding to the genus 0 leaves u_1, u_2, u_3, u_4 ,

- \mathfrak{s}_x is the top cluster X since $a_x = x\infty$,
- \mathfrak{s}_1 is an odd child of s_x of relative depth 2 since $a_1 = w_1w_x$ is blue of length 2,
- \mathfrak{s}_2 is an even non-übereven child of s_x of relative depth 1 since w_2 is blue and $a_2 = w_2w_x$ is yellow of length 2,
- \mathfrak{s}_3 is an übereven child of \mathfrak{s}_2 of relative depth 1 since w_3 is yellow and $a_3 = w_3w_2$ is yellow of length 2,
- \mathfrak{t}_1 and \mathfrak{t}_2 are children of \mathfrak{s}_x of relative depth $\frac{5}{2}$ since $a_{\mathfrak{t}_1} = u_1x$ and $a_{\mathfrak{t}_2} = u_2x$, both of length 5,
- \mathfrak{t}_3 and \mathfrak{t}_4 are children of \mathfrak{s}_3 of relative depth 3 since $a_{\mathfrak{t}_3} = u_3w_3$ and $a_{\mathfrak{t}_4} = u_4w_3$, both of length 6,
- \mathfrak{s}_1 has 5 roots since it is odd and of genus 2,
- \mathfrak{s}_3 has no roots outside $\mathfrak{t}_3, \mathfrak{t}_4$ since it is übereven,
- \mathfrak{s}_2 has 2 roots in addition to \mathfrak{s}_3 since it is even non-übereven of genus 0,
- \mathfrak{s}_x has 1 root in addition to $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}_1, \mathfrak{t}_2$ since it is even of genus 0.

This is precisely Σ .

The automorphism (α, ϵ) corresponding to (α_T, ϵ_T) is given by

- α swaps \mathfrak{t}_1 and \mathfrak{t}_2 and fixes all other clusters since α_T swaps u_1 and u_2 and fixes all other vertices,
- $\epsilon(X) = \epsilon(\mathfrak{t}_1) = \epsilon(\mathfrak{s}_2) = 1$, since ϵ_T is 1 on the parent edges of x, u_1 and w_2 ,
- $\epsilon(\mathfrak{t}_2) = \epsilon(\mathfrak{t}_3) = \epsilon(\mathfrak{t}_4) = \epsilon(\mathfrak{s}_3) = -1$, since ϵ_T is -1 on the parent edges of u_2, u_3, u_4, w_3 .

This yields $(\alpha, \epsilon) = (\alpha_\Sigma, \epsilon_\Sigma)$.

5. ONE-TO-ONE CORRESPONDENCE (CLOSED CASE)

In this section we study the notion of equivalence for open hyperelliptic graphs, open BY trees and cluster pictures (see Definitions 3.13, 3.26 and 3.43). This enables us to prove a ‘closed version’ of the correspondences of Section 4. We also

- Explain how to explicitly obtain the core of an open hyperelliptic graph or open BY tree, and address the converse, namely which open BY trees have a specified core (Proposition 5.7, Corollary 5.10, Table 5.6);
- Identify a canonical representative in an equivalence class of BY trees, that corresponds to a balanced cluster picture (Remark 5.15);
- Describe ‘principal clusters’ in a cluster picture, that correspond to vertices in the core of the associated hyperelliptic graph;
- Interpret the moves for equivalence of cluster pictures (Definition 3.43) in terms of the associated BY tree (proof of Lemma 5.20).

The precise statement of the closed correspondence is as follows:

Theorem 5.1. *There is a genus-preserving one-to-one correspondence, both in the metric and non-metric case, between*

- (i) *Balanced cluster pictures up to isomorphism,*
- (i') *Cluster pictures up to equivalence,*
- (ii) *Hyperelliptic graphs up to isomorphism,*
- (ii') *Open hyperelliptic graphs up to equivalence,*
- (iii) *BY trees up to isomorphism,*
- (iii') *Open BY trees up to equivalence.*

Explicitly, the correspondence between hyperelliptic graphs and BY trees is given by the maps \underline{G} and \underline{T} of Section 4 and similarly, the correspondence between open BY trees, open hyperelliptic graphs and cluster pictures is given by the maps $\underline{G}, \underline{T}$ and $\underline{\Sigma}$. Maps (ii') \rightarrow (ii) and (iii') \rightarrow (iii) are given by taking the core, whilst (i) \rightarrow (i') takes a balanced cluster picture to its equivalence class.

In genus ≥ 2 , the correspondences (i') \leftrightarrow (ii) \leftrightarrow (iii) set up bijections between various invariants⁸ as shown in Table 5.3.

Proof. The correspondence (ii) \leftrightarrow (iii) was shown previously in Proposition 4.11. Lemma 5.5 combined with Proposition 5.7 (both shown below) and the open correspondence of Theorem 4.2 gives (ii') \leftrightarrow (iii'). The correspondences (ii') \leftrightarrow (ii) and (iii') \leftrightarrow (iii) follow from the definition of equivalence: Proposition 5.7 shows that the core exists and is unique, and Corollary 5.10 and Remark 5.11 show that each closed hyperelliptic graph (resp. closed BY tree) arises as the core of some open hyperelliptic graph (resp. open BY tree). Proposition 5.21 below shows that two cluster pictures are equivalent if and only if the associated BY trees are, which combined with the open correspondence of Theorem 4.2 gives (i') \leftrightarrow (ii'). Finally, we show in Lemma 5.25 that each equivalence class of cluster pictures contains a unique balanced one, giving (i) \leftrightarrow (i').

The bijections between the invariants shown in Table 5.3 follow from the explicit description of the correspondences. \square

The above result makes no mention of automorphism groups. Since equivalent objects need not have the same automorphism groups, the situation is more delicate (see e.g. Examples 3.14, 3.28 and 3.44). However, the correspondence for balanced cluster pictures is fairly clean:

Theorem 5.2. *Let (X, Σ) be a balanced (possibly metric) cluster picture and \tilde{G} be the core of the associated hyperelliptic graph $\underline{G}(\Sigma)$. Then the natural map $\text{Aut}(\Sigma) \rightarrow \text{Aut}(\tilde{G})$, given by applying $\underline{G} \circ \underline{T}$ and restricting to the core, is surjective. The kernel is trivial if X is \bar{u} bereven, and C_2 if X is non- \bar{u} bereven (generated by the trivial permutation with $\epsilon(X) = -1$ and all other signs $+1$).*

⁸The definition of cotwins and principal clusters is given in Definition 5.17.

Proof. This follows from Corollary 5.26 along with the comparison of equivalence between hyperelliptic graphs and BY trees (Lemma 5.5 and Proposition 4.11). \square

5.1. Equivalence: BY trees, hyperelliptic graphs. Recall that the core of an open hyperelliptic graph (resp. open BY tree) is its maximal closed hyperelliptic subgraph (resp. closed BY subtree). Proposition 5.7 below shows that the core exists and is unique. Granted this, we single out vertices that come from the core:

Definition 5.4 (Principal vertex).

- (1) Let G be an open hyperelliptic graph of genus ≥ 2 , with core \tilde{G} . A vertex v of G is *principal* if it corresponds to a vertex in the core (i.e. lies in the core and does not become a point on an edge upon removing $G \setminus \tilde{G}$).
- (2) Let T be an open BY tree of genus ≥ 2 , with core \tilde{T} . A vertex v of T is *principal* if it corresponds to a vertex in the core which is not a genus 0 leaf of \tilde{T} .

Still assuming Proposition 5.7, we now show that cores and principal vertices are preserved under the correspondence:

Lemma 5.5. *The correspondences \underline{G} and \underline{T} between open hyperelliptic graphs and open BY trees preserve equivalence. Explicitly, if T is an open BY tree with core \tilde{T} , then the core of $\underline{G}(T)$ is isomorphic to $\underline{G}(\tilde{T})$. Similarly, for an open hyperelliptic graph G with core \tilde{G} , the core of $\underline{T}(G)$ is isomorphic to $\underline{T}(\tilde{G})$.*

Moreover, let G be an open hyperelliptic graph of genus ≥ 2 , $T = \underline{T}(G)$ the associated BY tree and $\pi : G \rightarrow T$ the quotient map. Then π induces a bijection between ι -orbits of principal vertices of G and principal vertices of T (here $\iota \in \text{Aut } G$ is the hyperelliptic involution).

Proof. Recall that the maps \underline{G} and \underline{T} of Constructions 4.4 and 4.8 were defined for both open and closed objects. It follows from the explicit constructions that if G is an open hyperelliptic graph and G' a (closed) hyperelliptic subgraph, then $\underline{T}(G')$ is canonically a closed BY subtree of $\underline{T}(G)$, and that inclusion of subtrees is preserved by \underline{T} . Since the same is also true if we start with an open BY tree and consider the map \underline{G} applied to subtrees of T , we obtain the result.

For the second part, it only remains to recall from Construction 4.4 that vertices of T either arise from ι -orbits of vertices of G , or from ι -anti-invariant edges, and that the latter are precisely the genus zero leaves of T . \square

TABLE 5.3. Dictionary for the one-to-one correspondence
(closed case, genus ≥ 2)

| Hyperelliptic graph | BY tree | Cluster picture |
|--|---|--|
| ι -invariant vertices of genus g | blue vertices of genus g that are not genus 0 leaves | principal non-übereven clusters of genus g |
| pairs of vertices $\{v, \iota(v)\}$ | yellow vertices of genus 0 | principal übereven clusters |
| loops of length 2δ | genus 0 leaves with the edge to a blue vertex of length 2δ | twins of distance δ to a principal non-übereven parent, and cotwins of distance δ to a non-übereven child, and X if $X = \mathfrak{s} \coprod \mathfrak{t}$ with \mathfrak{s} even non-übereven, \mathfrak{t} twin, of distance δ to each other |
| ι -anti-invariant edges of length 2δ , that are not loops | genus 0 leaves with the edge to a yellow vertex of length 2δ | twins of distance δ to a principal übereven parent, and cotwins of distance δ to an übereven child, and X if $X = \mathfrak{s} \coprod \mathfrak{t}$ with \mathfrak{s} übereven, \mathfrak{t} twin, of distance δ to each other |
| ι -invariant edges of length $\delta/2$ | blue edges of length δ | odd principal clusters of distance δ to a principal parent, and X if $X = \mathfrak{s}_1 \coprod \mathfrak{s}_2$ with $\mathfrak{s}_1, \mathfrak{s}_2$ odd principal of distance δ to each other |
| pairs of edges $\{e, \iota(e)\}$ of length δ | yellow edges of length 2δ not incident to a genus 0 leaf | even principal clusters of distance δ to a principal parent, and X if $X = \mathfrak{s}_1 \coprod \mathfrak{s}_2$ with $\mathfrak{s}_1, \mathfrak{s}_2$ even principal of distance δ to each other |

TABLE 5.6. Neighbourhoods of ∞ (genus ≥ 2)

| Case | Connected component of ∞ of $G \setminus \{\text{principal vertices}\}$ | Connected component of ∞ of $T \setminus \{\text{principal vertices}\}$ | Configuration of maximal principal clusters of Σ |
|------|---|---|--|
| O | | | |
| E | | | |
| U | | | |
| OE | | | |
| OU | | | |
| EE | | | |
| EU | | | |
| EO | | | |
| EOE | | | |
| EOU | | | |
| OxO | | | |
| ExE | | | |
| UxU | | | |
| UxE | | | |
| TxE | | | |
| TxU | | | |

In every row the three entries correspond to one another under $\underline{G}, \underline{T}, \underline{\Sigma}$.

Column 2/3: labelled vertices are principal, others are not.

Column 2/3: dashed edges, and the leftmost vertex in EO/EOE/EOU, are not in the core.

Column 4: O/E/U denotes an odd/even/übereven maximal principal cluster.

Proposition 5.7. *Let T be an open BY tree and G an open hyperelliptic graph.*

(1) *The core \tilde{T} of T (resp. \tilde{G} of G) exists and is unique.*

(2) *$g(\tilde{T}) = g(T)$ and $g(G) = g(\tilde{G})$.*

Now assume $g(T) \geq 2$ and $g(G) \geq 2$.

(3) *There are 16 possibilities for the connected component of ∞ of $T \setminus \{\text{principal vertices}\}$, and 16 corresponding ones for hyperelliptic graphs. They are given in Table 5.6.*

(4) *In the notation of Table 5.6, \tilde{T} is obtained from T by starting at ∞ and removing*

- (Cases O – EU) edge
- (Cases EO – EOU) edge, vertex, edge
- (Cases OxO – TxU) edge, vertex

and \tilde{G} from G by removing

- (Cases O – U) edge(s)
- (Case EO) edges, vertex, edge
- (Case EOE , EOU) edges, vertex, edge, vertex
- (Cases OE – EU , OxO – TxU) edge(s), vertex/vertices.

Here ‘edge/edges’ means removing the unique edge / ι -orbit of two edges from ∞ or the latest removed vertex; and ‘vertex/vertices’ means taking the vertex / ι -orbit of two vertices incident to the latest removed edge(s), and (a) when they have degree 1, removing them or (b) when they have degree 2, declaring them to not be vertices anymore (but interior points on the resulting merged edges instead).

Proof. That the core of T (resp. G) exists and is unique follows by inspection in genus 0 and 1 (see Tables 3.1, 4.1) and, otherwise, from the explicit construction of the core detailed below. For that, we just do the $T \rightarrow \tilde{T}$ case, the $G \rightarrow \tilde{G}$ case being its translation via the correspondence between open hyperelliptic graphs and open BY trees.

Clearly, to get from T to \tilde{T} , the unique open edge (say, from v_0) needs to be removed. If v_0 becomes a valid BY tree vertex (see Definition 3.18), we are done. There are 7 such configurations depending on whether v_0 is a genus 0 leaf or not, and on the edge colours (Cases O – EU).

Otherwise, after the open edge is removed, v_0 must violate 3.18 (1) or (2). If it violates (1), v_0 is yellow (of genus 0) with exactly two other edges. There are 5 such configurations depending on the two adjacent vertices, and on the edge colours (Cases ExE – TxU). Declaring v_0 to not be a vertex gives a BY tree.

If v_0 violates (2), it must have become a blue genus 0 vertex with no yellow edges. Then it has one or two blue edges (by 3.18 (3)), and the removed edge was yellow (by (2)). When there are two blue edges, this is case OxO . Declaring v_0 to not be a vertex gives a BY tree. If there is one blue edge, we remove v_0 and this edge — these are Cases EO , EOE , EOU depending on the vertex adjacent to v_0 .

The statement about the genus is clear, as no positive genus vertices are removed and the (relative) homology is unchanged. (See also Proposition 6.4.) \square

An immediate corollary is the following.

Corollary 5.8. *Let T be an open BY tree with core \tilde{T} . Then the map $\text{Aut } T \rightarrow \text{Aut } \tilde{T}$ given by restriction of automorphisms has kernel C_2 in Cases EE – OxO and is injective in all other cases. Its image consists of all those automorphisms of \tilde{T} fixing the point closest to ∞ .*

Remark 5.9. The proof of Proposition 5.7 gives a straightforward way to compute the core in practice.

We now consider the opposite direction. That is, given a closed BY tree \tilde{T} , which open BY trees have \tilde{T} as their core:

Corollary 5.10. *Let \tilde{T} be a closed BY tree. Then an open BY tree T has core \tilde{T} if and only if it is obtained from \tilde{T} in one of the following ways:*

- *declaring a point on an edge of \tilde{T} to be a vertex of genus 0 (and the same colour as the edge) and adding a yellow open edge at this vertex,*
- *adding a yellow open edge to a vertex of \tilde{T} ,*
- *adding a blue open edge to a blue vertex v of \tilde{T} which has $2g(v) + 2 > \# \text{blue edges at } v$,*
- *adding ‘closed blue edge \rightarrow genus 0 blue vertex \rightarrow open yellow edge’ to a blue vertex v of \tilde{T} which has $2g(v) + 2 > \# \text{blue edges at } v$.*

Proof. This is true by inspection in genus 0 and 1. Otherwise, it follows from Proposition 5.7 that the core of any open BY tree is obtained by removing either a single open edge, blue or yellow, or removing the configuration consisting of a closed blue edge followed by an open yellow edge. To prove the result, one just checks which conditions need to be satisfied at a point $x \in \tilde{T}$ in order for the graph given by glueing on one of these three configurations to be a valid open BY tree with core \tilde{T} . \square

Remark 5.11. Corollaries 5.8 and 5.10 have obvious analogues for hyperelliptic graphs via the correspondence. Since the statements are neater for BY trees and these are the ones we will use when comparing the notion of equivalence for BY trees/hyperelliptic graphs to that for cluster pictures, we have omitted them.

5.1.1. Centres of BY trees. Given a closed BY tree \tilde{T} , Corollary 5.10 can be viewed as describing the equivalence class of open BY trees with core (isomorphic to) \tilde{T} . In this subsection we single out a canonical representative in each equivalence class of open BY trees. To do this, we first single out a canonical ‘centre’ (either a vertex or edge) on a given closed BY tree. Glueing on an open yellow edge there gives the sought representative of the associated equivalence class of open BY trees.

The following purely graph theoretic lemma shows the existence of a ‘centre’ with respect to a weighting on the vertices of a tree. We omit the proof.

Lemma 5.12. *Let T be a finite connected tree and $w : V(T) \rightarrow \mathbb{R}_{\geq 0}$ be a ‘weight’ function on the vertices of T such that each vertex of degree one or two has positive weight. For a subtree $T' \leq T$, set $w(T') = \sum_{v \in T'} w(v)$ and for each $v \in T$, define*

$$\phi(v) = \max \{w(T') \mid T' \text{ is a connected component of } T \setminus \{v\}\}.$$

Then either

- (1) $\min_{v \in T} \phi(v) < \frac{1}{2}w(T)$, in which case the minimum is attained at a unique vertex of T , and all other vertices have $\phi(v) > \frac{1}{2}\phi(T)$,

or

- (2) $\min_{v \in T} \phi(v) = \frac{1}{2}w(T)$, in which case the minimum is attained at precisely two vertices of T , and these vertices are adjacent.

In case (1) we call the minimising vertex the centre of T with respect to the weighting ϕ . In case (2), we define the centre to be the edge joining the two minimising vertices.

Definition 5.13. Let T be a closed BY tree. We define its *centre* to be the vertex or edge afforded by Lemma 5.12 applied to the weight function $w : V(T) \rightarrow \mathbb{Z}_{\geq 0}$ given by

$$w(v) = \begin{cases} 0 & v \text{ yellow,} \\ 2(g) + 2 - \deg_{T_b}(v) & v \text{ blue,} \end{cases}$$

where here for a blue vertex v , $\deg_{T_b}(v)$ denotes the number of blue edges at v . Note that as w is invariant under all automorphisms of T , the centre of T is also.

Remark 5.14. Let T be a closed BY tree. With w as in Definition 5.13 above, the same argument as in the proof of Proposition 4.17 gives

$$w(T) = 2g(T) + 2.$$

Similarly, for each $v \in T$ and each connected component T' of $T \setminus \{v\}$ we have

$$w(T') = \begin{cases} 2g(T') + 2 & \text{if the open edge of } T' \text{ is yellow,} \\ 2g(T') + 1 & \text{if the open edge of } T' \text{ is blue.} \end{cases}$$

Remark 5.15. Glueing an open yellow edge to the centre of a closed BY tree \tilde{T} gives (up to isomorphism) a canonical representative in the equivalence class of open BY trees having \tilde{T} as their core. Letting T denote this representative, the natural map $\text{Aut } T \rightarrow \text{Aut } \tilde{T}$ given by restriction of automorphisms is surjective (this follows from Corollary 5.8 since all automorphisms of \tilde{T} fix the centre). The kernel of the restriction homomorphism is trivial if the centre of \tilde{T} is yellow, whilst if the centre of \tilde{T} is blue then the kernel is isomorphic to C_2 , generated by the automorphism of T which fixes all vertices, has sign -1 on (the component of T_y containing) the yellow open edge, and trivial sign on all other components of T_y (again see Corollary 5.8).

Remark 5.16. Since equivalence is preserved by the correspondence between open BY trees and open hyperelliptic graphs, the construction above gives a canonical representative in each equivalence class of open hyperelliptic graphs.

5.2. Equivalence: cluster pictures. We now turn to cluster pictures. We begin by describing the clusters that will correspond to principal vertices and to genus 0 leaves on the associated open BY tree,

5.2.1. *Principal clusters, twins and cotwins.*

Definition 5.17. Let (X, Σ) be a cluster picture of genus $g \geq 2$. Recall that a cluster of size 2 is a *twin*.

A cluster \mathfrak{s} is a *cotwin* if it has a child of size $2g$ whose complement is not a twin.

A proper cluster \mathfrak{s} is *principal* if it is neither a twin nor a cotwin, and if $|\mathfrak{s}| = 2g + 2$ then \mathfrak{s} has at least 3 children.

Example 5.18. Each of the 7 pictures from Example 3.44 has exactly one principal cluster (smallest one of size ≥ 4), and either one twin or one cotwin.

Remark 5.19. When $g(\Sigma) \geq 2$, either Σ has a unique maximal principal cluster \mathfrak{s} , of size $\geq 2g$, or $X = \mathfrak{s} \coprod \mathfrak{s}'$ is a union of two principal clusters. Marking a maximal principal cluster by O/E/U according to whether it is odd/even non-übereven/übereven gives 16 possible configurations, as listed in Table 5.6 (right column). Note that Σ has a cotwin if and only if it is in cases OE-EU or EOE-EOU.

Lemma 5.20. Let Σ be a cluster picture of genus ≥ 2 , and $T = \underline{T}(\Sigma)$ the associated open BY tree, with core \tilde{T} . Then a cluster $\mathfrak{s} \in \Sigma$ is principal if and only if the associated vertex $v_{\mathfrak{s}}$ of T (see Construction 4.13) is a principal vertex. Moreover, \mathfrak{s} corresponds to a genus 0 leaf in \tilde{T} if and only if it is either a twin or a cotwin.

Proof. This follows from Construction 4.13 (describing the association $\mathfrak{s} \mapsto v_{\mathfrak{s}}$) and Table 5.6, which shows what is removed to obtain the core. Note that twins correspond to genus 0 leaves of T , each of which remains a genus 0 leaf in the core, whilst cotwins correspond to vertices of T which are not genus 0 leaves but become so when passing to the core. \square

5.2.2. *Comparison with equivalence for open BY trees.* We now show that the maps \underline{T} and $\underline{\Sigma}$ between cluster pictures and open BY trees preserve equivalence. Since (up to isomorphism) these maps are inverse to each other, it suffices to show the result for \underline{T} .

Proposition 5.21. Two cluster pictures (X, Σ) , (X', Σ') are equivalent if and only if the corresponding BY trees $\underline{T}(\Sigma)$, $\underline{T}(\Sigma')$ are. The same holds in the metric case.

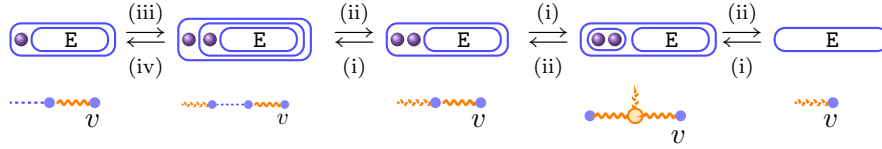
Proof. Again, this is true by inspection in genus 0 and 1 (see Tables 3.1, 4.1). Now suppose (X, Σ) is a cluster picture of genus ≥ 2 , and $T = \underline{T}(\Sigma)$.

In the notation of Table 5.6, cluster pictures without clusters of size $2g$ or $2g + 1$ fall into cases E, U, OxO, ExE, UxU and UxE. Possible moves (see Definitions 3.43, 3.45) between such cluster pictures are

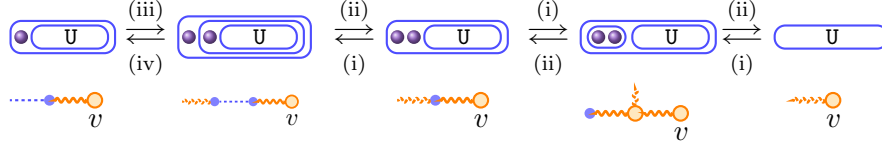
$$E/U \begin{array}{c} \xrightarrow{(i)} \\ \xleftarrow{(ii)} \end{array} O \times O / E \times E / U \times E / U \times U.$$

The corresponding BY tree in cases E and U has an open yellow edge attached to a principal vertex v . A principal child \mathfrak{s} of X corresponds to an adjacent principal vertex $v_{\mathfrak{s}}$ and the moves (i) above are obtained by adding a ‘cocluster’ to \mathfrak{s} . The effect on T is to move the yellow edge to a point on the edge between v and $v_{\mathfrak{s}}$ (without changing the metric on the core in the metric case). The moves (ii) above are the inverse of this.

The only moves to and from cluster pictures that have a cluster of size $2g$ are

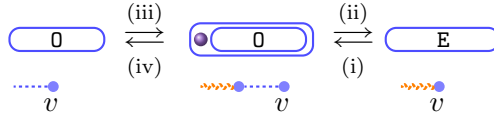


and



depending on whether the cluster of size $2g$ is non-übereven or übereven. By construction of T , moving along the chains transforms the ‘tail at ∞ ’ without altering the core, as shown above.

Finally, cluster pictures that have a cluster of size $2g + 1$ but not of size $2g$ are cases O and EO. The only moves between them are



As before, these transform the tail at ∞ without altering the core, as shown.

This covers all possible moves between cluster pictures. Thus, equivalent cluster pictures yield BY trees with isomorphic cores (in other words, equivalent).

Conversely, if T and T' are open BY trees with the same core, the moves described above, the fact that BY trees are connected and Corollary 5.10 show that the associated cluster pictures are equivalent. \square

Remark 5.22. Incidentally, the proof of the proposition shows that every equivalence of cluster pictures can be broken up into moves (i)-(iv) *uniquely* (without going back). More precisely, fix an equivalence class of cluster pictures. Consider the graph whose vertices are cluster pictures in this class (up to isomorphism) and edges are given by moves (i)-(iv). Then this graph is a tree. See Table 5.23 for an example; here directions of arrows indicate moves (ii) and (iv).

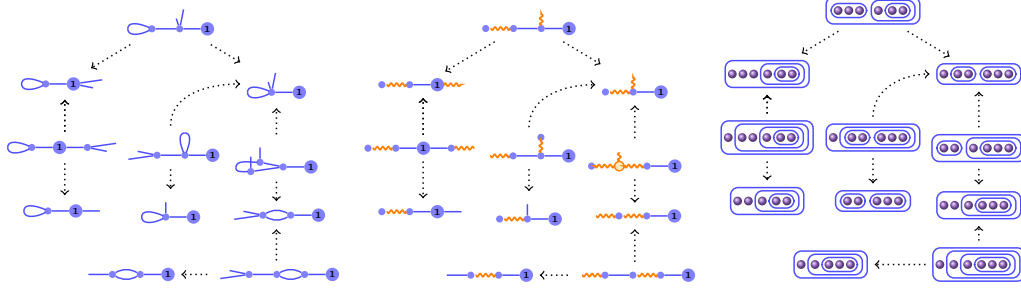


TABLE 5.23. Example: an equivalence class of hyperelliptic graphs, open BY trees, and cluster pictures (Type $1 \times I_n$)

Proposition 5.24. *Say that two proper clusters $\mathfrak{s}_1, \mathfrak{s}_2$ in a cluster picture (X, Σ) are adjacent if $\mathfrak{s}_1 < \mathfrak{s}_2$, $\mathfrak{s}_2 < \mathfrak{s}_1$, or $X = \mathfrak{s}_1 \amalg \mathfrak{s}_2$ with $\mathfrak{s}_1, \mathfrak{s}_2 < X$ and X is even. If (X, Σ) and (X', Σ') are equivalent cluster pictures of genus ≥ 2 , then there is an adjacency preserving bijection*

$$\{\text{principal } \mathfrak{s} \in \Sigma\} \leftrightarrow \{\text{principal } \mathfrak{s}' \in \Sigma'\},$$

$$\{\text{twins and cotwins } \mathfrak{s} \in \Sigma\} \leftrightarrow \{\text{twins and cotwins } \mathfrak{s}' \in \Sigma'\}.$$

In the metric case, we can also insist that the bijection preserves distances between clusters.

Proof. Let $T = \underline{T}(\Sigma)$ and let \tilde{T} be its core. By Lemma 5.20, under the map $\mathfrak{s} \mapsto v_{\mathfrak{s}}$, principal clusters of Σ correspond to vertices of \tilde{T} which are not genus 0 leaves, whilst twins and cotwins of Σ correspond to genus 0 leaves of \tilde{T} . First note that vertices $v_{\mathfrak{s}}$ and $v_{\mathfrak{s}'}$ are adjacent in the open BY tree T if and only if $\mathfrak{s} < \mathfrak{s}'$ or $\mathfrak{s}' < \mathfrak{s}$. That two vertices in the core \tilde{T} are adjacent if and only if the corresponding clusters are now follows by consulting Table 5.6. In particular, any isomorphism from \tilde{T} to the core of $\underline{T}(\Sigma')$ (one such necessarily exists by Proposition 5.21) induces a bijection as in the statement. \square

5.2.3. Centres and balanced cluster pictures. Recall that a cluster picture (X, Σ) is balanced if $|X| = 2g + 2$ is even, there are either 0 or 2 clusters of size $g + 1$, and X is the only cluster of size $> g + 1$. (For instance, in Table 4.1, the second row in each of the three groups is balanced.)

Lemma 5.25. *Every equivalence class of cluster pictures has (up to isomorphism) a unique balanced one. Under \underline{T} , it corresponds to the canonical representative of the associated equivalence class of open BY trees as defined in Remark 5.15.*

Proof. Let Σ be a cluster picture of genus g , $T = \underline{T}(\Sigma)$ the associated BY tree, and \tilde{T} its core. Then by Proposition 5.21, the cluster pictures

equivalent to Σ are precisely those associated to the open BY trees obtained from \tilde{T} by one of the operations of Corollary 5.10. Note that glueing an open blue edge to a vertex of \tilde{T} results in a cluster picture of odd size (see Table 5.6) and such cluster pictures are not balanced. Similarly, glueing a closed blue edge (whose endpoint is blue of genus 0) followed by an open yellow edge onto a vertex of \tilde{T} results in a cluster picture having a cotwin which again is not balanced.

Next, fix a vertex v of \tilde{T} and consider the cluster picture (X', Σ') associated to the open BY tree obtained by glueing a yellow open edge to v . We have $|X'| = 2g + 2$. Moreover, the children of X' are all of the form $\mathfrak{s}_{v'}$ for v' adjacent to v . For each such vertex, let $T_{v'}$ denote the connected component of $\tilde{T} \setminus v$ containing v' . By Remark 4.18, it follows that the size of $\mathfrak{s}_{v'}$ is equal to $w(T_{v'})$ where w is the weight function of Definition 5.13. It now follows from Lemma 5.12 that (X', Σ') is balanced if and only if v is the centre of \tilde{T} (see also Remark 5.14).

Similarly, one sees that the cluster picture associated to the open BY tree obtained by glueing an open yellow edge to an existing edge of \tilde{T} is balanced if and only if this edge is the centre of \tilde{T} . \square

Corollary 5.26. *Let (X, Σ) be a balanced cluster picture, let $T = \underline{T}(\Sigma)$ be the associated open BY tree and let \tilde{T} denote the core of T . Then the natural map $\text{Aut } \Sigma \rightarrow \text{Aut } \tilde{T}$, sending $\sigma \in \text{Aut } \Sigma$ to the restriction of $\underline{T}(\sigma)$ to \tilde{T} , is surjective. Its kernel is trivial if X is übereven, and C_2 if X is non-übereven (generated by the trivial permutation with $\epsilon(X) = -1$ and all other signs $+1$).*

Proof. By Theorem 4.2, the map $\text{Aut } \Sigma \rightarrow \text{Aut } T$ sending σ to $\underline{T}(\sigma)$ is an isomorphism. The result now follows from Lemma 5.25 and the corresponding statement for BY trees discussed in Remark 5.15. \square

6. THE HOMOLOGY LATTICE Λ

In this section we study the lattices Λ attached to hyperelliptic graphs, BY trees and cluster pictures, along with their natural automorphism actions, and show that the correspondences identify them.

Action of automorphisms. Recall how to identify automorphism groups across the correspondence. Suppose G is a hyperelliptic graph, $T = \underline{T}(G)$ the associated BY tree, and $\Sigma = \underline{\Sigma}(T)$ the associated cluster picture. A choice of a section $s : G/\langle \iota \rangle \rightarrow G$ as in Construction 4.4 gives an isomorphism $\text{Aut } G \rightarrow \text{Aut } T$, which makes Λ_T an $\text{Aut } G$ -module. There is also a canonical isomorphism $\text{Aut } T \rightarrow \text{Aut } \Sigma$, independent of any choices, which makes Λ_Σ into an $\text{Aut } G$ -module as well. Finally, automorphisms of an open BY tree T act on the core \tilde{T} , and similarly for hyperelliptic graphs.

Theorem 6.1 (Lattice correspondence). *If Σ is a cluster picture, then there are canonical $\text{Aut } \Sigma$ -equivariant isomorphisms*

$$\Lambda_{\widetilde{G(\underline{T}(\Sigma))}} \cong \Lambda_{G(\underline{T}(\Sigma))} \cong \Lambda_{\Sigma} \cong \Lambda_{\underline{T}(\Sigma)} \cong \Lambda_{\widetilde{\underline{T}(\Sigma)}}.$$

If T is a BY tree, then there are canonical $\text{Aut } T$ -equivariant isomorphisms

$$\Lambda_{\underline{G}(T)} \cong \Lambda_T \cong \Lambda_{\underline{\Sigma}(T)}.$$

If G is an open hyperelliptic graph, choose a section $s : G/\langle \iota \rangle \rightarrow G$. Then there are canonical $\text{Aut } G$ -equivariant isomorphisms

$$\Lambda_G \cong \Lambda_{\underline{T}(G)} \cong \Lambda_{\underline{\Sigma}(G)}.$$

For another section s' , the two isomorphisms $\Lambda_G \cong \Lambda_{\underline{T}(G)}$ differ by $\psi_{s,s'}$ of Proposition 4.7. The isomorphism $\Lambda_{\underline{T}(G)} \cong \Lambda_{\underline{\Sigma}(G)}$ does not depend on the choice of s .

In the metric case, all isomorphisms preserve the pairings.

Proof. This follows upon combining Lemmas 6.4 with Propositions 6.6 and 6.18 and Remark 6.7. \square

Corollary 6.2. *Let (X, Σ) and (X', Σ') be equivalent cluster pictures. Then there is an isomorphism $\Lambda_{\Sigma} \cong \Lambda_{\Sigma'}$ which, in the metric case, preserves the respective pairings.*

Proof. In both the metric and non-metric cases, the core \tilde{T} of $\underline{T}(\Sigma)$ is an invariant of its equivalence class. Hence so is the associated lattice $\Lambda_{\tilde{T}}$. The result now follows from Theorem 6.1. \square

Remark 6.3. A proof of Corollary 6.2 without passing through the correspondence can be given by using Proposition 5.24.

6.1. Reduction to the closed case. We begin by showing that the homology groups of open hyperelliptic graphs (resp. open BY trees) are isomorphic to those of their core. As a consequence we will only consider the closed case after this subsection.

Proposition 6.4. *Let G be an open hyperelliptic graph with core \tilde{G} . Then there is a canonical isomorphism*

$$H_1(G) \cong H_1(\tilde{G}),$$

equivariant for the action of $\text{Aut } G$. Similarly, if T is an open BY tree with core \tilde{T} then there is a canonical isomorphism

$$H_1(T, T_b) \cong H_1(\tilde{T}, \tilde{T}_b),$$

equivariant for the action of $\text{Aut } T$. (In the above, automorphisms of G (resp. T) act on $H_1(\tilde{G})$ (resp. $H_1(\tilde{T}, \tilde{T}_b)$) via their restriction to the core.)

Proof. It is easy to check the claim in genus 0 and 1 (see Tables 3.1, 4.1), so assume $g \geq 2$. From Table 5.6 we see that any open hyperelliptic graph admits a deformation retract onto its core. This induces the sought isomorphism on homology groups.

In the case of BY trees, Table 5.6 shows that T admits a deformation retract onto its core \tilde{T} which induces a deformation retract from T_b to \tilde{T}_b . This induces maps $H_i(T, T_b) \rightarrow H_i(\tilde{T}, \tilde{T}_b)$ for each i . It also induces maps $H_i(T) \rightarrow H_i(\tilde{T})$ and $H_i(T_b) \rightarrow H_i(\tilde{T}_b)$ for each i which, being induced by deformation retracts, are isomorphisms. That the maps on relative homology groups are also isomorphisms now follows from the relative homology exact sequence and the 5-lemma.

The claim about the action of automorphisms is immediate since the deformation retracts act as identity on the core by definition. \square

Remark 6.5. It follows from Lemma 6.4 that the action of an automorphism of an open hyperelliptic graph G (resp. open BY tree T) on $H_1(G)$ (resp. $H_1(T, T_b)$) depends only on its restriction to the core.

6.2. Hyperelliptic Graphs \leftrightarrow BY trees.

Proposition 6.6. *Let G be a hyperelliptic graph, $T = \underline{T}(G)$ the associated BY tree and $s : T \rightarrow G$ a section to the quotient map $\pi : G \rightarrow T$. Then there is a canonical isomorphism*

$$H_1(G) \cong H_1(T, T_b),$$

equivariant for the action of $\text{Aut } G$ and, in the metric case, preserving the respective pairings. (In the above, $\text{Aut } G$ acts on $H_1(T, T_b)$ via the isomorphism $\text{Aut } G \rightarrow \text{Aut } T$ determined by s (see Construction 4.4)).

Proof. We take the usual Δ -complex structure on T , so that the 0-simplices are the vertices and the 1-simplices are the edges. For the Δ -complex structure on G , we take the usual one, and then subdivide each ι -anti-invariant edge at the preimage of the associated vertex of T (which in each case is a genus 0 leaf). Define a map of complexes $C_\bullet(T) \rightarrow C_\bullet(G)$ given by $x \mapsto s(x) - \iota(s(x))$. Since the section s is continuous, this map is compatible with the boundary operators on each side (strictly speaking, we need to choose an orientation on the edges of T and G respectively to define the boundary operators; we do this in such a way that both ι and π are orientation-preserving). The kernel of this map of complexes is $C_\bullet(T_b)$ and, along with the quotient map $\pi : C_\bullet(G) \rightarrow C_\bullet(T)$ we obtain a short exact sequence of complexes

$$0 \rightarrow C_\bullet(T)/C_\bullet(T_b) \rightarrow C_\bullet(G) \rightarrow C_\bullet(T) \rightarrow 0.$$

Since $H_2(T) = 0 = H_1(T)$ (T is contractible) and $H_\bullet(T, T_b)$ is the homology of the leftmost complex, this sequence gives an isomorphism

$$H_1(T, T_b) \xrightarrow{\sim} H_1(G).$$

In the metric case, the pairings on $H_1(G)$ and $H_1(T, T_b)$ are induced by ones on $C_1(G)$ and $C_1(T)$ respectively. The scaling factors in Construction 4.4 are defined in such a way that the map in degree 1 in the short exact sequence

above preserves these. Similarly, the compatibility with automorphism actions can be checked on the level of the map $C_1(T) \rightarrow C_1(G)$ (again, see Construction 4.4). \square

Remark 6.7. For each section $s : T \rightarrow G$, write $f_s : H_1(G) \rightarrow H_1(T, T_b)$ for the (inverse of the) isomorphism constructed in Proposition 6.6. Then given two sections s and s' one has

$$f_{s'} = \psi_{s,s'} \circ f_s$$

where $\psi_{s,s'} \in \text{Aut } T$ is as in Definition 4.6.

Remark 6.8. Another approach to proving the existence of the isomorphism of Proposition 6.6 is as follows. Writing G_b for the subgraph of G fixed by the hyperelliptic involution, one has $H_1(G) \cong \tilde{H}_0(G_b)$. This follows by enlarging the closed sets $T_b \subset T, G_b \subset G$ to their small open neighbourhoods \tilde{T}_b, \tilde{G}_b and applying the Mayer–Vietoris sequence to the open sets $U = \tilde{G}_b \cup s(T_y)$ and $V = \tilde{G}_b \cup \iota(s(T_y))$ which cover G (note that $U \cap V$ is homotopic to G_b , whilst U and V individually are homotopic to the tree T). Since G_b and T_b are homeomorphic, we have $\tilde{H}_0(G_b) \cong \tilde{H}_0(T_b)$. The latter group is isomorphic to $H_1(T, T_b)$ via the relative homology sequence.

6.3. BY trees. In this subsection we give an explicit description of the first relative homology group of a (closed, possibly metric) BY tree with respect to its blue part. This will be necessary for establishing the second isomorphism of Theorem 6.1 but may also be of independent interest. It will be convenient to work with rooted BY trees, i.e. BY trees with a distinguished point (which may be a vertex but could also be a point on an edge). Our description of the relative homology group will naturally be compatible with automorphisms of the BY tree which fix the root (but not general automorphisms).

6.3.1. Rooted BY trees.

Definition 6.9. By a *rooted BY tree* we mean a pair (T, R) where T is a (closed, possibly metric) BY tree and R , the ‘root’, is a point on T (i.e. a vertex or a point on an edge). By an automorphism of a rooted BY tree we mean an automorphism of the underlying BY tree (complete with signs on yellow components) that preserves R . We write $\text{Aut}_R T$ for the group of automorphisms of a rooted BY tree (T, R) . Given a vertex $v \neq R$ of T , we refer to the unique edge of v in the direction of R as the *parent edge* of v .

Remark 6.10. Every BY tree has a centre (in the sense of Definition 5.13), which is fixed by all automorphisms. Thus any BY tree can be made into a rooted BY tree in such a way that there is no difference between ‘rooted’ and ‘non-rooted’ automorphisms.

Definition 6.11. Let T be an open BY tree and \tilde{T} its core. Then we give \tilde{T} the structure of a rooted BY tree by defining the root R to be the point on \tilde{T} which is closest to ∞ in T .

Remark 6.12. Let \tilde{T} be the core of an open BY tree T of genus ≥ 2 . Then whether or not the root R is a vertex depends on the type of the neighbourhood of infinity in T (cf. Table 5.6). Specifically, it is a vertex of \tilde{T} in cases O–EOU, and lies on an edge otherwise. In cases O–E and OE–EOU, R is a blue vertex and in case U , R is a yellow vertex. In case OxO, R is a point on a blue edge and in cases ExE–TxU, R is a point on a yellow edge. Note that the automorphism group of an open BY tree \tilde{T} maps surjectively to the automorphism group of its core, viewed as a rooted BY tree (the map being restriction of automorphisms). This is an isomorphism apart from cases EE–OxO, where the kernel is isomorphic to C_2 and acts trivially on $H_1(T, T_b)$ (see Corollary 5.8).

6.3.2. Homology of a rooted BY tree. Let (T, R) be a rooted (closed) BY tree with blue part T_b and yellow part T_y . For the purposes of computation, we make T into a Δ -complex in the usual way, save that, in the case that the root R lies on an edge, we subdivide this edge at the root R so that R becomes a 0-simplex. Moreover, we orient all edges so that they point towards R . Note that every automorphism of T is orientation preserving since it fixes the root.

For the rest of the section we adopt the following convention.

Convention 6.13. If T' is a closed subtree of a rooted BY tree (T, R) then we take as the root of T' the point on T' closest to R in T (which is either R itself or a vertex of both T and T' (or both)).

Definition 6.14. Let (T, R) be a rooted BY tree. For a connected component Y of T_y , let \bar{Y} denote its closure in T , viewed as a rooted tree with root R_Y as in Convention 6.13. Write \mathcal{L}_Y for the set of (non-root) leaves of \bar{Y} and define $\mathcal{L}_T = \cup_Y \mathcal{L}_Y$ (note that this union is disjoint); equivalently,

$$\mathcal{L}_T = \{\text{blue vertices } v \neq R \text{ whose parent edge is yellow}\}.$$

For $v \in \mathcal{L}_T$, take Y for which $v \in \mathcal{L}_Y$ and define $\hat{v} = R_Y$. We then define the free \mathbb{Z} -module Π_T by

$$\Pi_T = \begin{cases} \mathbb{Z}[\mathcal{L}_T] & \text{if } R \text{ is blue,} \\ \left\{ \sum_v \lambda_v v \in \mathbb{Z}[\mathcal{L}_T] \mid \sum_{\hat{v}=R} \lambda_v = 0 \right\} & \text{if } R \text{ is yellow.} \end{cases}$$

In the metric case, we define a pairing on $\mathbb{Z}[\mathcal{L}_T]$ and Π_T by setting

$$\langle v_1, v_2 \rangle = \begin{cases} \delta(v_1 \wedge v_2, \hat{v}_1) & \text{if } \hat{v}_1 = \hat{v}_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $v_1 \wedge v_2$ denotes the point (vertex or R (or both)) at which the unique paths in T from v_1 to R and v_2 to R meet. Automorphisms $(\sigma, \epsilon_\sigma) \in \text{Aut}_R T$ act on $\mathbb{Z}[\mathcal{L}_T]$ and Π_T by setting, for a leaf $v_1 \in \bar{Y}$,

$$(\sigma, \epsilon_\sigma) \cdot v_1 = \epsilon_\sigma(Y) \sigma(v_1),$$

and extending linearly.

Remark 6.15. For $v \in \mathcal{L}_T$, \hat{v} is blue unless R is yellow and $\hat{v} = R$ (as all yellow vertices of T have only yellow edges).

Remark 6.16. Note that the action of $\text{Aut}_R T$ on $\mathbb{Z}[\mathcal{L}_T]$ is particularly simple, being given by signed permutations.

Proposition 6.17. *Let (T, R) be a rooted BY tree. Then there is a canonical isomorphism*

$$\Pi_T \cong H_1(T, T_b),$$

equivariant for the action of $\text{Aut}_R T$ and, in the metric case, respecting the pairings.

Proof. First note that the map sending a connected component of T_b to its root gives a bijection between the set of connected components of T_b not containing R and the set \mathcal{L}_T . It now follows that

$$\text{rk } \Pi_T = \text{rk } H_1(T, T_b),$$

as Remark 3.24 shows that the rank of $H_1(T, T_b)$ is one less than the number of connected components of T_b .

Now consider the homomorphism $p : \mathbb{Z}[\mathcal{L}_T] \rightarrow C_1(T)/C_1(T_b)$ where for $v \in \mathcal{L}$, we define $p(v)$ as the shortest path in T from v to \hat{v} . Since this path is yellow by construction, p is injective. Note also that $C_1(T)/C_1(T_b)$ is a free \mathbb{Z} -module since $C_1(T)$ is the direct sum of $C_1(T_b)$ and $C_1(T_y)$. We claim that the image of p is a direct summand of $C_1(T)/C_1(T_b) = C_1(T_y)$. Indeed, for each $v \in \mathcal{L}_T$, its parent edge appears in $p(v)$ with multiplicity one, and does not appear in $p(v')$ for any $v \neq v' \in \mathcal{L}_T$. Thus the set $\{p(v) \mid v \in \mathcal{L}_T\}$ may be completed to a basis for $C_1(T_y)$ by adding in all yellow edges of T except the parent edges of vertices in \mathcal{L}_T .

Now denote by \tilde{p} the restriction of p to Π_T , which is injective since p is. We claim that its image is contained in $H_1(T, T_b)$. Indeed, writing $d : C_1(T) \rightarrow C_0(T)$ for the boundary map as usual, for $v \in \mathcal{L}_T$ we have $d(p(v)) = \hat{v} - v$. Provided that \hat{v} is blue (i.e. unless R is yellow and $\hat{v} = R$), we see that $d(p(v))$ lies in $C_0(T_b)$ in which case $p(v)$ is an element of $H_1(T, T_b)$. In particular, the claim holds for R blue. On the other hand, if R is yellow and $v, v' \in \mathcal{L}_T$ with $\hat{v} = \hat{v}' = R$, then $d(p(v - v')) = v' - v \in C_0(T_b)$. By the way we have defined Π_T when R is yellow, this proves the claim in this instance also.

Since Π_T is a direct summand of $\mathbb{Z}[\mathcal{L}_T]$ (it is the kernel of the homomorphism into \mathbb{Z} sending $v \in \mathcal{L}_T$ with \hat{v} yellow to 1, and all other elements of \mathcal{L}_T to 0, which shows that $\mathbb{Z}[\mathcal{L}_T]/\Pi_T$ is torsion free), p is injective, and $p(\mathbb{Z}[\mathcal{L}_T])$ is a direct summand on $C_1(T_y)$, it follows that $\tilde{p}(\Pi_T)$ is a direct summand of $C_1(T_y)$ also. We are now in the following situation: we have inclusions of free, finite rank \mathbb{Z} -modules $\tilde{p}(\Pi_T) \subseteq H_1(T, T_b) \subseteq C_1(T_y)$, with $\tilde{p}(\Pi_T)$ a direct summand of $C_1(T_y)$ and $\text{rk } \tilde{p}(\Pi_T) = \text{rk } H_1(T, T_b)$. It now follows formally that $\tilde{p}(\Pi_T) = H_1(T, T_b)$ whence \tilde{p} is an isomorphism. \square

6.4. Cluster Pictures \leftrightarrow BY trees.

Proposition 6.18. *Let (X, Σ) be a cluster picture, $T = \underline{T}(\Sigma)$ the associated open BY tree and \tilde{T} its core. Then there is a canonical isomorphism*

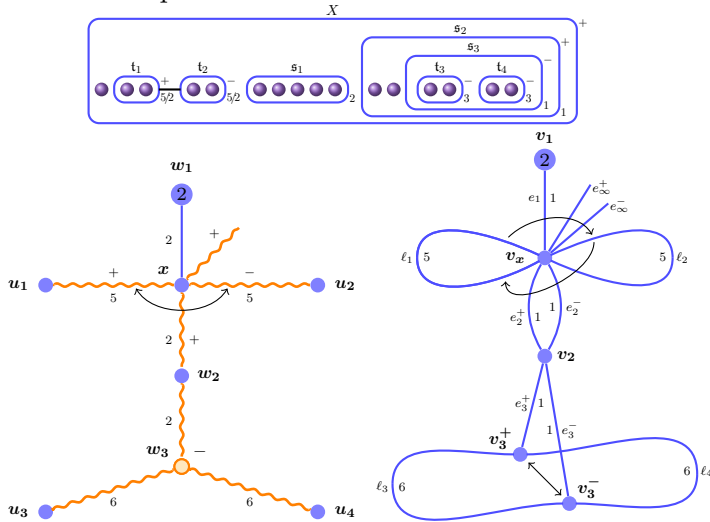
$$\Lambda_\Sigma \cong \Pi_{\tilde{T}},$$

equivariant for the action of $\text{Aut } \Sigma = \text{Aut } T$ and, in the metric case, preserving the respective pairings. (Here, Π_T is as defined in Definition 6.14.)

Proof. We first claim that, under the correspondence between cluster pictures and open BY trees, the set \mathcal{E}_Σ of Definition 3.47 (consisting of even, non-übereven clusters $\mathfrak{s} \neq X$) is identified with the set $\mathcal{L}_{\tilde{T}}$ (see Definition 6.14). Indeed, under the correspondence, even non-übereven clusters correspond to blue vertices of T whose parent edge is yellow. Now if \mathfrak{s} is an even non-übereven cluster with associated vertex $v_{\mathfrak{s}}$, one easily checks from Table 5.6 that the parent edge is in the core if and only if $\mathfrak{s} \neq X$. Thus, as desired, \mathcal{E}_Σ corresponds to the set of blue vertices in the core which are not the root, and whose parent edge is yellow.

It now follows that the map $\mathfrak{s} \leftrightarrow v_{\mathfrak{s}}$ induces an isomorphism between Λ_Σ and $\Pi_{\tilde{T}}$. Moreover, given $\mathfrak{s} \in \mathcal{E}_\Sigma$, corresponding to a leaf $v_{\mathfrak{s}}$ of a yellow component Y of \tilde{T} , one sees that $\hat{\mathfrak{s}}$ (cf. Definition 3.47) corresponds to the root R_Y of \tilde{Y} . Indeed, this is immediate if \mathfrak{s} is contained in some non-übereven cluster, and the case where no such cluster exists follows upon consulting Table 5.6. The claimed result now follows immediately from the definitions of Λ_Σ and $\Pi_{\tilde{T}}$, complete with pairing and action of automorphisms (using the identification of automorphism groups as given in Proposition 4.19). \square

6.5. An example. Consider Examples 3.49, 3.32, 3.17, 4.21 and 4.22 with T and G open. We construct bases for $\Lambda_\Sigma, \Lambda_T, \Lambda_G$ that illustrate why the lattices are isomorphic.



Recall that by definition

$$\Lambda_\Sigma = \mathbb{Z}\mathbf{t}_1 + \mathbb{Z}\mathbf{t}_2 + \mathbb{Z}\mathbf{t}_3 + \mathbb{Z}\mathbf{t}_4 + \mathbb{Z}\mathbf{s}_2,$$

the basis vectors corresponding to the even, non-übereven clusters in Σ .

We produce a basis for $\Lambda_T = H_1(T, T_b)$ by looking at one yellow component at a time. For a given yellow component Y (with closure \bar{Y}), let z be the leaf of \bar{Y} closest to ∞ and as basis vectors take the shortest paths from the other leaves to z . In our example, we have 4 yellow components and the basis vectors are $[u_1, x]$, $[u_2, x]$, $[u_3, w_2]$, $[u_4, w_2]$, $[w_2, x]$, which is precisely the basis in Example 3.32. In other words, the basis we've chosen is indexed by blue vertices with a yellow edge towards ∞ , which exactly correspond to even, non übereven clusters of Σ (see Example 4.22).

To see why the two pairings coincide, consider for example the two paths $[u_3, w_2]$, $[u_4, w_2]$. Their intersection is of length 2 which agrees with the distance $\delta(w_3, w_2)$. Since w_2 corresponds to $\mathbf{s}_2 = \hat{\mathbf{t}}_3 = \hat{\mathbf{t}}_4$ and w_3 corresponds to $\mathbf{s}_3 = \mathbf{t}_3 \wedge \mathbf{t}_4$,

$$2 = \langle [u_3, w_2], [u_4, w_2] \rangle = \delta(w_3, w_2) = \delta(\mathbf{t}_3 \wedge \mathbf{t}_4, \hat{\mathbf{t}}_3) = \langle \mathbf{t}_3, \mathbf{t}_4 \rangle = 2.$$

Now consider the action of the automorphisms $\sigma_\Sigma = (\alpha_\Sigma, \epsilon_\Sigma)$ and $\sigma_T = (\alpha_T, \epsilon_T)$ on the lattices. Recall that $\sigma_\Sigma(\mathbf{s}) = \epsilon_\Sigma(\mathbf{s})\alpha_\Sigma(\mathbf{s})$ by definition, e.g. $\sigma_\Sigma(\mathbf{t}_2) = -\mathbf{t}_1$ and that $\sigma_T([u, v]) = \epsilon_T(z)[\alpha_T(u), \alpha_T(v)]$, where $[u, v]$ is a path within one yellow component and z any point on that component, e.g. $\sigma_T([u_2, x]) = -[u_1, x]$.

A path in our basis of the form $[u, *]$ is sent to $[\alpha_T(u), *]$ which is another basis vector, e.g. $\alpha_T([u_2, x]) = [u_1, x]$. By construction, if u corresponds to the cluster \mathbf{s} then $\alpha_T(u)$ corresponds to $\alpha_\Sigma(\mathbf{s})$ and $\epsilon_T(u) = \epsilon_\Sigma(\mathbf{s})$, e.g. $\epsilon_T([u_2, x]) = -1 = \epsilon_\Sigma(\mathbf{t}_2)$. It follows that the action of σ_T on Λ_T is the same as the action of σ_Σ on Λ_Σ .

Now consider the hyperelliptic graph G with the decomposition of its ι -permuted part $G_y = G_y^+ \amalg G_y^-$, where G_y^+ consists of $e_2^+, e_3^+, v_3^+, e_\infty^+$ and the top halves of ℓ_1, ℓ_2, ℓ_3 and ℓ_4 (call these $\ell_1^+, \ell_2^+, \ell_3^+, \ell_4^+$). To construct a basis for Λ_G in a systematic way

- construct G' from G_y by removing its edges towards ∞ and taking the closure. In our example $G' = G \setminus \{e_1, v_1, e_\infty^+, e_\infty^-\}$,
- the ι -invariant points remaining are v_x, v_2 and the mid-points of $\ell_1, \ell_2, \ell_3, \ell_4$,
- for ι -invariant points with an edge in G' towards ∞ , create a loop by following G_y^+ towards ∞ to the next ι -invariant point and back via G_y^- ; for our example we obtain the loops ℓ_1 (oriented clockwise) and ℓ_2 (anti-clockwise), and the loops $e_2^+ - e_2^-, \ell_3^+ + e_3^+ - e_3^- - \ell_3^-, \ell_4^+ + e_3^+ - e_3^- - \ell_4^-$, where we have oriented each edge and half-edge towards ∞ . It is exactly the basis given in Example 3.17.

Under the $2 : 1$ map $G \rightarrow T$, these loops correspond to yellow paths from one blue vertex to another blue vertex which is closer to ∞ . By construction, this gives the basis of Λ_T .

Since both pairings measure the length of the intersection of loops/paths, we get the same pairing on both spaces, e.g.

$$\begin{aligned} 2 &= \langle [u_3, w_2], [u_4, w_2] \rangle = \delta(w_3, w_2) = 2\delta(e_3^+) = \delta(e_3^+) + \delta(e_3^-) = \\ &= \langle \ell_3^+ + e_3^+ - e_3^- - \ell_3^-, \ell_4^+ + e_3^+ - e_3^- - \ell_4^- \rangle = 2. \end{aligned}$$

As in Example 4.21, the action of σ_G on $G/\langle \iota \rangle = T$ is that of α_T , in particular $\alpha_T([u_2, x]) = [u_1, x]$ corresponds to $\sigma_G(\ell_2) = \pm \ell_1$. Moreover $\epsilon_T([u_2, x]) = -1$ corresponds to $\sigma_G(\ell_2^+) = \ell_1^-$ so that $\sigma_T([u_2, x]) = -[u_1, x]$ corresponds precisely to $\sigma_G(\ell_2) = -\ell_1$.

7. TAMAGAWA GROUPS OF HYPERELLIPTIC GRAPHS

In this section we study the Tamagawa group $\Phi(G)$ (see Definition 7.4) of a hyperelliptic graph G whose edge lengths are integers, and the corresponding group for BY trees and cluster pictures. In Proposition 7.10 we identify it with the graph-theoretic Jacobian of G along with automorphism action. We then give an explicit description of the 2-torsion in this group (Corollary 7.13).

7.1. Integral hyperelliptic graphs and Tamagawa groups.

Definition 7.1. A (closed) metric hyperelliptic graph G is *integral* if all edge lengths are integers, unless G is the genus 1 circle graph from (3.3). In that exceptional case, we say that G is integral if the sum of lengths of its two edges is an integer.

A closed metric BY tree T is *integral* if $\underline{G}(T)$ is.

A metric cluster picture (X, Σ) is *integral* if the core of $\underline{G}(\underline{T}(\Sigma))$ is.

Lemma 7.2. *A (closed) metric BY tree is integral if and only if all edges have integral length and all edges not incident to a genus 0 leaf have even length.*

A metric cluster picture (X, Σ) of genus ≥ 2 is integral if and only if

- $\delta(\mathfrak{s}, \mathfrak{s}') \in 2\mathbb{Z}$ for $\mathfrak{s}' < \mathfrak{s}$ with $\mathfrak{s}', \mathfrak{s}$ principal, \mathfrak{s}' odd,
- $\delta(\mathfrak{s}, \mathfrak{s}') \in 2\mathbb{Z}$ for $\mathfrak{s}, \mathfrak{s}'$ odd principal, $X = \mathfrak{s} \sqcup \mathfrak{s}'$,
- $\delta(\mathfrak{s}, \mathfrak{s}') \in \mathbb{Z}$ for $\mathfrak{s}' < \mathfrak{s}$ with $\mathfrak{s}', \mathfrak{s}$ principal, \mathfrak{s}' even,
- $\delta(\mathfrak{s}, \mathfrak{s}') \in \mathbb{Z}$ for $\mathfrak{s}, \mathfrak{s}'$ even principal, $X = \mathfrak{s} \sqcup \mathfrak{s}'$,
- $\delta(\mathfrak{s}, \mathfrak{t}) \in \frac{1}{2}\mathbb{Z}$ for a twin $\mathfrak{t} < \mathfrak{s}$,
- $\delta(\mathfrak{s}, \mathfrak{c}) \in \frac{1}{2}\mathbb{Z}$ for \mathfrak{c} a cotwin, $\mathfrak{s} < \mathfrak{c}$ of size $2g$.

Proof. This follows immediately from the (closed) case of the correspondence between cluster pictures, BY trees and hyperelliptic graphs as detailed in Section 5 (see in particular Table 5.3). \square

Remark 7.3. Given an integral hyperelliptic graph G , the pairing on $H_1(G, \mathbb{Z})$ takes integer values. In particular, $H_1(G, \mathbb{Z})$ embeds in its abstract dual $H_1(G, \mathbb{Z})^\vee$ via $x \mapsto \langle x, - \rangle$. By using the correspondences of previous sections, it follows that if X is either an integral BY tree or an integral cluster picture, then Λ_X embeds into its abstract dual Λ_X^\vee similarly.

Definition 7.4. Let X be a hyperelliptic graph/BY tree/cluster picture and suppose that X is integral. Then we define the *Tamagawa group* $\Phi(X)$ as

$$\Phi(X) = \Lambda_X^\vee / \Lambda_X.$$

In each case, the action of $\text{Aut } X$ on Λ_X induces an action on $\Phi(X)$.

Theorem 7.5 (Tamagawa group correspondence). *Let (X, Σ) be an integral cluster picture, T (resp. G) the associated open BY tree (resp. open hyperelliptic graph) and \tilde{T} (resp. \tilde{G}) its core. Then we have isomorphisms*

$$\Phi(\Sigma) \cong \Phi(\tilde{T}) \cong \Phi(\tilde{G}),$$

equivariant for the action of $\text{Aut } \Sigma$.

Proof. This follows immediately from Theorem 6.1. \square

7.2. Jacobians of graphs. In this section, we show that the Tamagawa group of an integral hyperelliptic graph G coincides with the Jacobian of a (combinatorial) graph $G_{\mathbb{Z}}$ canonically associated to G . Throughout this subsection, G has genus ≥ 2 .

Notation 7.6. For an integral hyperelliptic graph G , we denote by $G_{\mathbb{Z}}$ the graph having the same underlying topological space as G , but whose set of vertices consists of those points on G which are an integer distance from the vertices of G . Equivalently, $G_{\mathbb{Z}}$ is the graph obtained by subdividing each edge e of G , say of length l , by adding $l - 1$ vertices at intervals of unit distance along the edge, so as to obtain a new graph all of whose edge lengths are 1.

Remark 7.7. We have $\text{Aut } G_{\mathbb{Z}} = \text{Aut } G$ and the discussion in Section 2.2.4 shows that $H_1(G_{\mathbb{Z}})$ is canonically isomorphic to $H_1(G)$, with the isomorphism preserving the respective pairings and automorphism actions.

In what follows we shall think of $G_{\mathbb{Z}}$ as being a finite combinatorial graph with unweighted edges (though possibly with loops and multiple edges) and disregard the genus marking. We now recall the definition of the Jacobian of such a graph.

Definition 7.8. Let \mathcal{G} be a finite combinatorial graph, possibly with loops and multiple edges (we reserve the letter ‘ G ’ for hyperelliptic graphs). Write $\text{Div}(\mathcal{G})$ for the free \mathbb{Z} -module on the vertices $V(\mathcal{G})$ of \mathcal{G} and $\text{Div}^0(\mathcal{G})$ for the subgroup of $\text{Div}(\mathcal{G})$ consisting of elements whose coefficients sum to zero. Contained in $\text{Div}^0(\mathcal{G})$ is a certain full rank submodule $\text{Prin}(\mathcal{G})$ consisting of ‘principal divisors’ which may be defined as follows:

For $v, v' \in V(\mathcal{G})$, set

$$v \cdot v' = \begin{cases} \deg(v) - 2 \text{ \# loops at } v & v = v', \\ -\text{\# edges between } v \text{ and } v' & v \neq v', \end{cases}$$

and define a map $\alpha : \text{Div}(\mathcal{G}) \rightarrow \text{Div}(\mathcal{G})$ by, for $v \in V(\mathcal{G})$, setting

$$\alpha(v) = \sum_{v' \in V(\mathcal{G})} (v \cdot v')v',$$

and extending linearly. We then have $\text{Prin}(\mathcal{G}) = \text{im}(\alpha)$.

The *Jacobian* of \mathcal{G} is then defined as

$$\text{Jac}(\mathcal{G}) = \text{Div}^0(\mathcal{G}) / \text{Prin}(\mathcal{G}).$$

It is a finite abelian group and the action of $\text{Aut } \mathcal{G}$ on $\text{Div}(\mathcal{G})$ induces an action on $\text{Jac}(\mathcal{G})$.

Remark 7.9. The notion of the Jacobian of a graph appears in multiple places in the literature and is referred to by several different names, the most notable other ones being the sandpile group and the Picard group (see [17, Section 1.1] and the references therein for an overview of its occurrence). Various equivalent definitions of the Jacobian also appear in the literature. The definition above is a slight variant of the one given in [4]. There the Jacobian is only defined for graphs without loops (but possibly with multiple edges). Our definition of $v \cdot v'$ above ensures that our definition of $\text{Jac}(\mathcal{G})$ (along with automorphism action) agrees with that of the Jacobian of the graph obtained by removing all loop-edges from \mathcal{G} .

We also remark that in [5, Section 3.1] a generalisation of the Jacobian is defined for arbitrary metric graphs. In the case that G is an integral hyperelliptic graph the group $\text{Jac}_{\mathbb{Z}}(G)$ in the notation of loc. cit. agrees with $\text{Jac}(G_{\mathbb{Z}})$ as defined above. However, since the definition of the Jacobian of a metric graph is less elementary than that of a finite combinatorial graph, we have elected to work with $G_{\mathbb{Z}}$ rather than introduce $\text{Jac}_{\mathbb{Z}}(G)$.

Proposition 7.10. *Let G be an integral hyperelliptic graph of genus ≥ 2 . Then there is a canonical isomorphism*

$$\text{Jac}(G_{\mathbb{Z}}) \cong \Phi(G),$$

equivariant for the action of $\text{Aut } G_{\mathbb{Z}} = \text{Aut } G$.

Remark 7.11. Several versions of this proposition, in various levels of generality (in particular, it is not specific to hyperelliptic graphs), appear in the literature though to the best of our knowledge the action of automorphism groups is not considered. We begin by reducing to the situation covered by [4, Theorem B.4] and deduce the compatibility of automorphisms from the explicit map defined there.

Proof of Proposition 7.10. When defining $\Phi(G)$ as $H_1(G)^{\vee} / H_1(G)$ we are at liberty to choose the Δ -complex structure on G and we do so by taking the

0-simplices to consist of the vertices of G and the 1-simplices as the edges of $G_{\mathbb{Z}}$ (along with their endpoints). Note that if e is a loop-edge of $G_{\mathbb{Z}}$ then it generates an orthogonal direct summand of $H_1(G)$ and, having length 1, we have $\langle e, e \rangle = 1$. In particular we see that e does not contribute to the quotient $H_1(G)^{\vee}/H_1(G)$. Combining this observation with Remark 7.9, it suffices to prove the result under the assumption that $G_{\mathbb{Z}}$ contains no loops.

We are now in the situation covered by [4, Theorem B.4] and our choice of Δ -complex structure on G ensures that $H_1(G)$, as computed with this choice, coincides with their $\Lambda^1(G_{\mathbb{Z}})$. Following loc. cit., we now fix a base vertex $v \in V(G_{\mathbb{Z}})$ and define a map

$$f_v : \text{Div}(G_{\mathbb{Z}}) \rightarrow H_1(G)^{\vee}/H_1(G) = \Phi(G)$$

as follows. Given $v' \in V(G_{\mathbb{Z}})$, pick a path $p_{v,v'}$ in $G_{\mathbb{Z}}$ from v to v' and set $f_v(v') = \langle p_{v,v'}, - \rangle$. Since $p_{v,v'}$ has integral length, its pairing with all elements of $H_1(G)$ is integral and so it defines a valid element of $H_1(G)^{\vee}$. Moreover, given two different choices of path from v to v' , their difference is an element of $H_1(G)$ so f_v is independent of the choice of path $p_{v,v'}$. Restricting f_v to $\text{Div}^0(G_{\mathbb{Z}})$ we obtain a map $f : \text{Div}^0(G_{\mathbb{Z}}) \rightarrow \Phi(G)$ which does not depend on the choice of base vertex v . Then as asserted in loc. cit. (see [2, Proposition 7.2] for the proof), the map f induces the sought isomorphism $\text{Jac}(G_{\mathbb{Z}}) \cong \Phi(G)$.

With the explicit map in hand, it is easy to check compatibility with automorphisms. Let $\sigma \in \text{Aut } G$ and view it as an automorphism of $G_{\mathbb{Z}}$. Let $v, v' \in V(G_{\mathbb{Z}})$. Then f sends $v - v' \in \text{Div}^0(G_{\mathbb{Z}})$ to $\langle p_{v,v'}, - \rangle$ where $p_{v,v'}$ is any path from v to v' . Now $\sigma(p_{v,v'})$ is a path from $\sigma(v)$ to $\sigma(v')$ and $\sigma(v) - \sigma(v') \in \text{Div}^0(G_{\mathbb{Z}})$ is mapped by f to $\langle \sigma(p_{v,v'}), - \rangle$, which is the same as we obtain by acting by σ on $f(v - v')$. The result now follows since $\text{Div}^0(G_{\mathbb{Z}})$ is generated by the elements $v - v'$ as v and v' range over the vertices of $G_{\mathbb{Z}}$. \square

7.3. 2-torsion in the Tamagawa group. As an application of the correspondence between hyperelliptic graphs and BY trees, and the description of the group $H_1(T, T_b)$ for a BY tree T afforded by Proposition 6.17, we end this section by computing the 2-torsion in the Tamagawa group of a hyperelliptic graph.

The result for BY trees is the following.

Theorem 7.12. *Let T be an integral BY tree of genus ≥ 2 . Write S for the set of connected components of T_b , excluding the genus 0 leaves of T whose unique (necessarily yellow) edge has odd length; $\text{Aut } T$ acts naturally on S . Then, as an $\text{Aut } T$ -module,*

$$\Phi(T)[2] \cong \begin{cases} 0 & S = \emptyset \text{ and } \text{rk } H_1(T, T_b) \text{ even,} \\ \mathbb{Z}/2\mathbb{Z} & S = \emptyset \text{ and } \text{rk } H_1(T, T_b) \text{ odd,} \\ \ker \left((\mathbb{Z}/2\mathbb{Z})[S] \xrightarrow{\text{sum}} \mathbb{Z}/2\mathbb{Z} \right) & \text{else,} \end{cases}$$

where ‘sum’ denotes the sum of the coefficients map.

Proof. For the time being we will ignore the action of $\text{Aut } T$, adding it back in at the end. Let $R \in T$ be a vertex and make T into a rooted BY tree by taking R to be the root. Now let Π_T be as in Definition 6.14, so that by Proposition 6.17 we have an isomorphism of \mathbb{Z} -lattices $\Pi_T \cong H_1(T, T_b)$ and, in particular, we have

$$\Phi(T) \cong \Pi_T^\vee / \Pi_T.$$

Noting that Π_T is torsion free as a \mathbb{Z} -module and applying the snake lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Pi_T & \longrightarrow & \Pi_T^\vee & \longrightarrow & \Phi(T) \longrightarrow 0 \\ & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ 0 & \longrightarrow & \Pi_T & \longrightarrow & \Pi_T^\vee & \longrightarrow & \Phi(T) \longrightarrow 0, \end{array}$$

it follows that we have

$$\Phi(T)[2] \cong \ker (\Pi_T / 2\Pi_T \longrightarrow \Pi_T^\vee / 2\Pi_T^\vee).$$

Suppose first that $S \neq \emptyset$, so that T has either a blue vertex which is not a genus 0 leaf, or that T has a genus 0 leaf whose unique edge has even length, and take R to be one such. Then since R is blue, we have $\Pi_T = \mathbb{Z}[\mathcal{L}_T]$ where the set \mathcal{L}_T of Definition 6.14 (which depends on R) consists of the blue vertices different from R whose parent edge is yellow. The pairing on Π_T is given by

$$\langle v_1, v_2 \rangle = \begin{cases} 0 & \widehat{v}_1 \neq \widehat{v}_2, \\ \delta(v_1 \wedge v_2, \widehat{v}_1) & \text{else,} \end{cases}$$

for $v_1, v_2 \in \mathcal{L}_T$ (see Definition 6.14 for the definitions of \widehat{v} and $v_1 \wedge v_2$).

We claim that for any $v_1, v_2 \in \mathcal{L}_T$ we have

$$\langle v_1, v_2 \rangle \equiv \begin{cases} 0 \pmod{2} & v_1 \neq v_2, \\ l_p(v) \pmod{2} & v_1 = v = v_2, \end{cases}$$

where for $v \in T$ (not equal to R) $l_p(v)$ is the length of its parent edge. To prove the claim, first take $v_1 \neq v_2 \in \mathcal{L}_T$ and assume that $\widehat{v}_1 = \widehat{v}_2$ (otherwise v_1 pairs trivially with v_2 by definition and we are done). Then $v_1 \wedge v_2$ cannot be a leaf and so each (necessarily yellow) edge on the shortest path from $v_1 \wedge v_2$ to \widehat{v}_1 has even length. It now follows from Lemma 7.2 that $\delta(v_1 \wedge v_2, \widehat{v}_1)$ is an even integer as desired. The case where $v_1 = v_2$ is similar: every edge in the path from v_1 to \widehat{v}_1 is a yellow edge not incident to a genus 0 leaf, save possibly for the parent edge of v_1 .

Now for $v \in \mathcal{L}_T$, let ϕ_v denote the homomorphism in Π_T^\vee dual to v (i.e. sending v to 1 and all other elements of \mathcal{L}_T to 0). Then by the claim, we see that the map $\Pi_T / 2\Pi_T \rightarrow \Pi_T^\vee / 2\Pi_T^\vee$ is given by $v \mapsto l_p(v)\phi_v$. Since the set $\{\phi_v | v \in \mathcal{L}_T\}$ is a basis for Π_T^\vee (the dual basis to the standard basis for $\Pi_T = \mathbb{Z}[\mathcal{L}_T]$), the kernel of this map is the \mathbb{F}_2 -vector space having as basis

the elements $v \in \mathcal{L}_T$ for which $l_p(v)$ is even. Now by Lemma 7.2, $v \in \mathcal{L}_T$ can only have $l_p(v)$ odd if it is a genus 0 leaf. In particular, writing O for the set of genus 0 leaves in T whose unique edge has odd length, an \mathbb{F}_2 -basis for $\ker(\Pi_T/2\Pi_T \rightarrow \Pi_T^\vee/2\Pi_T^\vee)$ is given by the set $\mathcal{L}_T \setminus O$. The map sending $v \in \mathcal{L}_T$ to its connected component in T_b is a bijection onto the set of connected components of T_b not containing R . It follows that the isomorphism of the statement in the case $S \neq \emptyset$ holds abstractly.

To additionally obtain the isomorphism as $\text{Aut } T$ -modules, recall from the proof of Proposition 6.17 that the canonical isomorphism of \mathbb{Z} -lattices $\Pi_T \cong H_1(T, T_b)$ is given by sending $v \in \mathcal{L}_T$ to the unique shortest path $p(v)$ between v and \hat{v} . It follows from the argument above that, as $\text{Aut } T$ -modules, $\Phi(T)[2]$ is isomorphic to the subgroup of $H_1(T, T_b)/2H_1(T, T_b)$ generated by the set $\{p(v) \mid v \in \mathcal{L}_T \setminus O\}$. One checks that the isomorphism $H_1(T, T_b) \rightarrow \tilde{H}_0(T_b)$ coming from the relative homology sequence (Remark 3.24) identifies this subgroup with $\ker\left((\mathbb{Z}/2\mathbb{Z})[S] \xrightarrow{\text{sum}} \mathbb{Z}/2\mathbb{Z}\right)$. Since the map $H_1(T, T_b) \rightarrow \tilde{H}_0(T_b)$ is $\text{Aut } T$ -equivariant upon passing to quotients by multiplication by 2 on each side (since then we no longer need to consider orientation or signs) we are done.

Suppose now that $S = \emptyset$, so that all blue vertices of T are genus 0 leaves and each of their edges has odd length. Then T necessarily has a yellow vertex and now we take the root R to be one such. Note that now \mathcal{L}_T is precisely the set of genus 0 leaves of T . Now T necessarily has precisely one yellow component, whence Π_T sits in a short exact sequence

$$0 \rightarrow \Pi_T \rightarrow \mathbb{Z}[\mathcal{L}_T] \xrightarrow{\text{sum}} \mathbb{Z} \rightarrow 0,$$

the map ‘sum’ sending $\sum_{v \in \mathcal{L}_T} \lambda_v v$ to the sum of the λ_v . Since Π_T is a free \mathbb{Z} -module, the sequence remains exact after applying the functor $\text{Hom}(-, \mathbb{Z})$ (which we denote $(-)^{\vee}$ for simplicity) and we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Pi_T & \longrightarrow & \mathbb{Z}[\mathcal{L}_T] & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \Pi_T^{\vee} & \longleftarrow & (\mathbb{Z}[\mathcal{L}_T])^{\vee} & \longleftarrow & \mathbb{Z}^{\vee} \longleftarrow 0 \end{array}$$

where here the two vertical maps are induced by the pairing (see Definition 6.14). Since each object in the diagram is torsion free, tensoring by $\mathbb{Z}/2\mathbb{Z}$ we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Pi_T/2\Pi_T & \longrightarrow & \mathbb{Z}[\mathcal{L}_T]/2\mathbb{Z}[\mathcal{L}_T] & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \Pi_T^{\vee}/2\Pi_T^{\vee} & \longleftarrow & (\mathbb{Z}[\mathcal{L}_T])^{\vee}/2(\mathbb{Z}[\mathcal{L}_T])^{\vee} & \longleftarrow & \mathbb{Z}^{\vee}/2\mathbb{Z}^{\vee} \longleftarrow 0. \end{array}$$

The same argument as in the case $S \neq \emptyset$ shows that the rightmost of the two vertical maps sends $v \in \mathcal{L}_T$ to its dual vector ϕ_v (each $l_p(v)$ being odd)

and as such is injective. Moreover, the map $\mathbb{Z}^\vee/2\mathbb{Z}^\vee \rightarrow (\mathbb{Z}[\mathcal{L}_T])^\vee/2(\mathbb{Z}[\mathcal{L}_T])^\vee$ sends the unique non-trivial element of $\mathbb{Z}^\vee/2\mathbb{Z}^\vee$ to $\sum_{v \in \mathcal{L}_T} \phi_v$. Combining exactness in the middle of the bottom row with the injectivity of the rightmost vertical map shows that $\sum_{v \in \mathcal{L}_T} v$ is the unique non-trivial element of the kernel of the map $\mathbb{Z}[\mathcal{L}_T]/2\mathbb{Z}[\mathcal{L}_T] \rightarrow \Pi_T^\vee/2\Pi_T^\vee$ given by composing the rightmost vertical map with the restriction map $(\mathbb{Z}[\mathcal{L}_T])^\vee/2(\mathbb{Z}[\mathcal{L}_T])^\vee \rightarrow \Pi_T^\vee/2\Pi_T^\vee$. Further, the top sequence shows that $\sum_{v \in \mathcal{L}_T} v$ lies in $\Pi_T/2\Pi_T$ if and only if $|\mathcal{L}_T| = \text{rk } H_1(T, T_b) + 1$ is even. Thus

$$\ker(\Pi_T/2\Pi_T \rightarrow \Pi_T^\vee/2\Pi_T^\vee) \cong \begin{cases} 0 & \text{rk } H_1(T, T_b) \text{ even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{rk } H_1(T, T_b) \text{ odd,} \end{cases}$$

which completes the proof of the theorem (note that we do not need to consider the action of $\text{Aut } T$ in this case since the only possible action of any group on $\mathbb{Z}/2\mathbb{Z}$ is trivial). \square

Corollary 7.13. *Let G be a hyperelliptic graph of genus ≥ 2 . Write G_b for the subgraph of G fixed by the hyperelliptic involution and write $G(\mathbb{Z})$ for the set of points on G which are an integer distance from a vertex. Let \mathcal{W} denote the set of connected components of G_b which contain a point of $G(\mathbb{Z})$. Then we have isomorphisms of $\text{Aut } G$ -modules*

$$\Phi(G)[2] \cong \begin{cases} 0 & \mathcal{W} = \emptyset \text{ and } \text{rk } H_1(G) \text{ even,} \\ \mathbb{Z}/2\mathbb{Z} & \mathcal{W} = \emptyset \text{ and } \text{rk } H_1(G) \text{ odd,} \\ \ker\left((\mathbb{Z}/2\mathbb{Z})[\mathcal{W}] \xrightarrow{\text{sum}} \mathbb{Z}/2\mathbb{Z}\right) & \text{else,} \end{cases}$$

where ‘sum’ denotes the sum of the coefficients map.

Proof. Let $T = \underline{T}(G)$ be the BY tree associated to G . Then the quotient map gives a homeomorphism from G_b to T_b . Let Z be a connected component of T_b . Then as yellow vertices of T have only yellow edges, Z necessarily contains a vertex of T . In fact, either Z contains a vertex which is not a genus 0 leaf, or $Z = \{v\}$ for a single genus 0 leaf v . In the first case, the preimage under π of this vertex is a vertex of $\pi^{-1}(Z)$. On the other hand, if $Z = \{v\}$ for a genus 0 leaf v , then $\pi^{-1}(v)$ is not a vertex of G but the midpoint of an ι -anti-invariant edge. In particular, $\pi^{-1}(v) \in G(\mathbb{Z})$ if and only if the parent edge of v has even length. Thus, the set \mathcal{W} corresponds under π to the set of connected components of T_b excluding the genus 0 leaves whose parent edge has odd length. Since the number of components of T_b is equal to $\text{rk } H_1(T, T_b) - 1$ (see Remark 3.24), the result now follows from Theorem 7.12, along with usual identification of $\text{Aut } G$ with $\text{Aut } T$ (choices of section here are irrelevant since all signs act trivially on the objects involved). \square

8. CLASSIFICATION OF SEMISTABLE TYPES AND NAMING CONVENTION

8.1. Types of BY trees (and hyperelliptic graphs/cluster pictures).

We propose a naming scheme for cluster pictures, (open) BY trees and (open) hyperelliptic graphs. We define it for BY trees and transport to the other two categories via the one-to-one correspondence.

Notation 8.1. Let T be an *open BY tree*. For the edges, we use


| | |
|--------------------|--------------------|
| \cdot | blue edge |
| $:$ | yellow edge |
| $\cdot_{d/2}, :_d$ | edge of length d |

and for the vertices

| | |
|------------------|---------------------------------------|
| U | yellow vertex |
| I or 0 | blue vertex of genus 0 |
| $1, 2, 3, \dots$ | blue vertex of genus $1, 2, 3, \dots$ |

To define the notation (‘Type’) of T itself, let e be its open edge, say incident to a vertex v . As a topological space, T decomposes as a disjoint union

$$T = \{v\} \cup \{e\} \cup t_1 \cup \dots \cup t_k \cup T_1 \cup \dots \cup T_n.$$

where the t_i are open trees ‘’ (a blue vertex of genus 0 with one open yellow edge, say of length n_i), and the T_i are the other connected components of $T \setminus \{e, v\}$; they are open BY trees. Then we define, inductively,

$$\text{Type}(T) = [e][v]_{n_1, \dots, n_k} \text{Type}(T_1) \cdots \text{Type}(T_n),$$

where $[e]$ is the notation for the edge e , as above, $[v]$ is the notation for the vertex v , as above. To avoid ambiguity, when $n > 0$, unless T is the full tree that we are interested in, we bracket everything after $[e]$ and write

$$\text{Type}(T) = [e]([v]_{n_1, \dots, n_k} \text{Type}(T_1) \cdots \text{Type}(T_n)).$$

In the non-metric case, the subscripts n_i are placeholders instead of lengths whose purpose is only to record the number of genus 0 leaves⁹. See Example 8.3 below.









Notation 8.2. For a *closed BY tree* \tilde{T} , recall from Remark 5.15 that there is a canonical open BY tree T with core \tilde{T} obtained by glueing a yellow open edge e_0 to the centre of \tilde{T} . We let $\text{Type}(\tilde{T})$ to be the name of T with $[e_0]$ omitted. To emphasize the configurations for which the centre is an edge, we also use an alternative notation

$$\begin{aligned} 0 \cdot_m \text{Type}(T_1) \cdot_n \text{Type}(T_2) &\longmapsto \text{Type}(T_1) \times_{m+n} \text{Type}(T_2) \\ U \cdot_m \text{Type}(T_1) :_n \text{Type}(T_2) &\longmapsto \text{Type}(T_1) \circ_{m+n} \text{Type}(T_2). \end{aligned}$$

The symbol \times or \circ can only appear once in the name, so the names of T_1 and T_2 do not have to be bracketed.

Example 8.3. Here are a few examples:

⁹this agrees with Kodaira and Namikawa-Ueno types for semistable curves of genus 1 and 2

| T (open) | Type(T) | \tilde{T} (closed) | Type(\tilde{T}) | genus |
|---|--------------------|---|---------------------|-------|
|  | $\cdot 1$ |  | 1 | 1 |
|  | $:U_{n,m,k}$ |  | $U_{n,m,k}$ | 2 |
|  | $\cdot 0:1$ |  | 1_n | 2 |
|  | $:0:(0 \cdot I_n)$ |  | $I_n \times I_m$ | 2 |

Example 8.4. In genus 2 the 7 balanced configurations (Table 9.5) are

$$2 \quad 1_n \quad I_{n,m} \quad U_{n,m,k} \quad 1 \times 1 \quad 1 \times I_n \quad I_n \times I_m$$

Example 8.5. The BY tree in Example 3.32 has Type $:0_{5,5} \cdot 12:2(0:2U_{6,6})$ and its core Type $0_{5,5} \cdot 12:2(0:2U_{6,6})$.

Remark 8.6. All the main invariants of a BY tree T can be seen from the type name: vertices which are not genus 0 leaves are the capitals I, U, 0, 1, 2, ... in the name. The genus 0 leaves are their subscripts; say there are k of them. Edges not incident to genus 0 leaves are the symbols \cdot , $:$, \times and \circ . The genus of T is the sum of all (non-subscript) numbers in the type plus

$$\text{rk } \Lambda_T = \# \text{colons} + k - \# \text{'U's} - \# \text{'o's}$$

and similarly for open BY trees, by ignoring the first symbol if it is ‘:’.

8.2. Automorphisms.




Notation 8.7 (Automorphism). Let T be a (possibly open) BY tree, and $E = \{e_1, \dots, e_n\} \subset E(T)$ an ordered subset of its edges, preserved by $\text{Aut } T$ (or some subgroup that we care about). By default, we take

$$E = \{\text{all closed edges}\},$$

ordered as follows: if $e = \{v_1, v_2\}$, $e' = \{v'_1, v'_2\}$, we check which $v \in \{v_1, v_2, v'_1, v'_2\}$ comes last in the name Type(T) (as a capital letter or its subscript); if it is v'_i then e' comes after e , and vice versa.

We then write σ as a permutation on the indices of the blue edges and \pm indices of the yellow edges, where the sign of $\pm\sigma(e)$ is determined by $\epsilon(e)$.

Example 8.8. Take a BY tree of genus 2 that corresponds to two nodal genus 0 curves meeting at a point, with two nodes of the same depth:

| | | | | |
|------------------|---|---|--|------------|
| Type | T | $\underline{\Sigma}(T)$ | $\underline{G}(T)$ | Aut(T) |
| $I_n \times I_n$ |  |  |  | D_4 |

Its automorphism group is D_4 (order 8). To write its elements we order the edges as above:

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ 1 \quad 2 \quad 3 \end{array}$$

As signed permutations, the elements of $\text{Aut } T$ are

$$\begin{aligned} & \text{id}, (-1 \ 1), (-3 \ 3), (-1 \ 1)(-3 \ 3), \\ & (13)(-1 \ -3), (1 \ -3)(-1 \ 3), (1 \ -3 \ -1 \ 3), (1 \ 3 \ -1 \ -3) \end{aligned}$$

On the corresponding hyperelliptic graph



the element $(-1\ 1)$ reflects the left loop in the x -axis, $(-3\ 3)$ reflects the right loop, and $(1\ 3\ -1\ -3)$ sends the left loop to the right one keeping the orientation and the right one to the right one reversing the orientation. (In this example, it is also reasonable to take $E = \{\text{yellow edges}\}$ instead of all edges.)

8.3. BY trees with an automorphism. For semistable hyperelliptic curves over local fields, it is important to keep track of the action of Frobenius. Therefore we need a naming convention for BY trees with a distinguished automorphism.

Notation 8.9 (Type with an automorphism). We incorporate the action of an automorphism on the edges and the signs into the type name, as follows.

Suppose T is an open BY tree, and $\phi \in \text{Aut } T$. As in Notation 8.1, write

$$T = \{v\} \cup \{e\} \cup t_1 \cup \dots \cup t_k \cup T_1 \cup \dots \cup T_n.$$

We extend the notation

$$\text{Type}(T) = [e][v]_{n_1, \dots, n_k} \text{Type}(T_1) \cdots \text{Type}(T_n).$$

to a notation for $\text{Type}(T, \phi)$ as follows:

The automorphism ϕ permutes the t_i and the T_i , and we rearrange them, if necessary, by ϕ -orbits. In other words, each ϕ -orbit, say of length m , is a block t_i, \dots, t_{i+m-1} or T_i, \dots, T_{i+m-1} .

- (1) For each ϕ -orbit t_i, \dots, t_{i+m-1} replace commas in n_i, \dots, n_{i+m-1} by \sim , a symbol for ‘are in the same ϕ -orbit’.
- (2) Similarly for each ϕ -orbit $\text{Type}(T_i) \cdots \text{Type}(T_{i+m-1})$, let ϕ^k be the smallest power of ϕ that stabilises T_i . Instead of $\text{Type}(T_j)$ write $\text{Type}(T_j, \phi^k)$, defined inductively, with the first edge symbol ‘.’ or ‘:’ replaced by \sim for $j > i$.
- (3) For a closed BY tree (see Notation 8.2), we similarly replace \times, \circ by \times, \circ when the endpoints of the central edge are swapped by ϕ .

We decorate the type with signs as follows. For a vertex v , let ϕ^{k_v} be the smallest power of ϕ that stabilizes v and ϵ_v be the sign of ϕ^{k_v} on its parent edge should it be yellow. For each vertex v that is first in its ϕ -orbit (ordered by appearance in the type):

- if v is yellow such that its parent edge does not lead to a yellow vertex, decorate the symbol for v in the type name with the superscript ϵ_v ,
- if v is blue, let w_1, \dots, w_s be the blue vertices joined to v by a yellow edge leading away from ∞ (ordered by appearance in the type). For each w_i that is first in its ϕ^{k_v} -orbit, decorate the symbol for v in the type name with the superscript ϵ_{w_i} . By convention, these superscripts appear in the same order as the w_i ’s and are separated by commas.

Finally if the open edge is yellow and incident to a blue vertex, decorate the initial colon with the sign of ϕ on the open edge.

In the case of a closed BY tree \tilde{T} with an automorphism ϕ , define the type (\tilde{T}, ϕ) to be the type (T, ϕ') with the first dot or colon (and their sign) deleted, where T is obtained from \tilde{T} by glueing a yellow open edge to its centre, and ϕ' extends ϕ . We use an analogous convention as in Notation 8.1 for the cases where the centre is an edge. In these cases, we decorate \circ with the sign of the initial U and we write $\tilde{\circ}, \tilde{\times}$ if T_1 and T_2 are swapped by ϕ .

One can check that, in the open or closed case, (T, ϕ) and (T', ϕ') get the same notation if and only if they are isomorphic as pairs, that is there is an isomorphism $\psi : T \rightarrow T'$ such that $\psi \circ \phi = \phi' \circ \psi$.

Example 8.10 (Elliptic curves). Let T be one of the BY trees



The associated cluster pictures are all possible ones of size 3 (see Table 4.1)



and they correspond to elliptic curves with good and multiplicative reduction. If $\phi \in \text{Aut } T$, then $\text{Type}(T, \phi)$ is $\cdot 1$ in the first case, and $\cdot I_n^+, \cdot I_n^-$ in the second case, depending on the ϕ -action on the yellow edge. When ϕ is Frobenius, $\cdot I_n^+$ is ‘split multiplicative’ and $\cdot I_n^-$ ‘non-split multiplicative’ reduction. If one is only interested in elliptic curves and not general curves of genus 1, one could omit the first dot and write the types as $1, I_n^+$ and I_n^- .

Example 8.11 ($I_n \times I_n$). In Example 8.8, for the 5 conjugacy classes of automorphisms $\phi \in D_4 = \text{Aut } T$ the label $\text{Type}(T, \phi)$ is

$$I_n^+ \times I_n^+, \quad I_n^+ \times I_n^-, \quad I_n^- \times I_n^-, \quad I_n^+ \times I_n, \quad I_n^- \times I_n.$$

See Table 9.6 for all possible types with an automorphism in genus 2.

Example 8.12. The BY tree with automorphism from Example 3.32 has type $:^+ 0_{5-5}^- :^+ 12 : 2(0 : 2 U_{6,6}^-)$

9. CLASSIFICATION FOR SMALL GENERA

Recall that in genus 0, 1 and 2 there are, respectively, 1, 2 and 7 ‘semistable types’, that is equivalence classes of hyperelliptic graphs/BY trees/cluster pictures (cf. Theorem 5.1 and Table 3.1 p. 11). It is easy to classify them in any genus g by listing all balanced cluster pictures in $X = \{1, \dots, 2g+2\}$, up to S_{2g+2} -conjugacy and then using Theorem 5.1 to obtain the corresponding hyperelliptic graphs and BY trees. The classification in genus 3 is given by the following theorem.

Theorem 9.1. *There are 32 hyperelliptic graphs of genus 3 up to isomorphism, 32 BY trees of genus 3 up to isomorphism and 32 cluster pictures of genus 3 up to equivalence. The one-to-one correspondence of Theorem 5.1 is given in Table 9.4.*

We can similarly classify all open hyperelliptic graphs by listing all possible cluster pictures. This is given in Table 4.1 (p. 22) for genus 0 and 1. For genus 2 the complete classification is the following:

Theorem 9.2. *There are 45 isomorphism classes of genus 2 open hyperelliptic graphs, open BY trees and cluster pictures. The one-to-one correspondence of Theorem 4.2 is given in Table 9.5.*

In the arithmetic setting of Section 1.3 one often wants to know the hyperelliptic graph together with an automorphism. In the context of curves over local fields of odd residue characteristic, these correspond to all possible Frobenius actions on the dual graph of the special fiber of the minimal regular model of semistable genus 2 curves. For elliptic curves, the corresponding types are 1, I_n^+ and I_n^- — good, split multiplicative and non-split multiplicative reduction; see Example 8.10.

Note that by Theorems 5.1 and 5.2, there is a bijection between

- Isomorphism classes of pairs (Σ, ϕ) , where Σ is a balanced cluster picture and $\phi \in \text{Aut } \Sigma$ has sign +1 on X if X is non-übereven. Here two pairs (Σ, ϕ) and (Σ', ϕ') are isomorphic if there is an isomorphism $\psi : \Sigma \rightarrow \Sigma'$ such that $\psi\phi\psi^{-1} = \phi'$.
- Isomorphism classes of pairs (G, ϕ) of hyperelliptic graphs with an automorphism, where two pairs (G, ϕ) and (G', ϕ') are isomorphic if there is an isomorphism $\psi : G \rightarrow G'$ such that $\psi\phi\psi^{-1} = \phi'$.

Explicitly, the bijection is given by mapping Σ to the core G of $\underline{G}(\Sigma)$ and ϕ to the restriction of $\underline{G}(\phi)$ to G . This makes it easy to list the types on the level of cluster pictures. The lattice Λ , the ϕ -action on it and the Tamagawa group can also be computed from it. (Definition 3.48).

Theorem 9.3. *Up to isomorphism, there are 23 genus 2 hyperelliptic graphs together with an automorphism. These are given in Table 9.6 together with their corresponding BY trees, balanced cluster pictures and lattices Λ .*

| Type name | G | T | Σ |
|---------------------------------|-----|-----|----------|
| 3 | | | |
| 2_n | | | |
| $1_{n,m}$ | | | |
| $I_{n,m,k}$ | | | |
| $U_{n,m,k,l}$ | | | |
| $2 \cdot I_n$ | | | |
| $2 \cdot 1$ | | | |
| $1_n \cdot I_m$ | | | |
| $1_n \cdot 1$ | | | |
| $0_{n,m} \cdot I_k$ | | | |
| $0_{n,m} \cdot 1$ | | | |
| $1 \cdot I_n \cdot I_m$ | | | |
| $1 \cdot 1 \cdot I_n$ | | | |
| $1 \cdot 1 \cdot 1$ | | | |
| $0_n \cdot I_m \cdot I_k$ | | | |
| $0_n \cdot 1 \cdot I_m$ | | | |
| $0_n \cdot 1 \cdot 1$ | | | |
| $0 \cdot 1 \circ 0 \cdot 1$ | | | |
| $0 \cdot I_n \circ 0 \cdot 1$ | | | |
| $0 \cdot I_n \circ 0 \cdot I_m$ | | | |
| $U_{n,m} \circ 0 \cdot 1$ | | | |
| $U_{n,m} \circ 0 \cdot I_k$ | | | |
| $U_{n,m} \circ U_{k,l}$ | | | |
| $I_n \circ 0 \cdot 1$ | | | |
| $I_n \circ 0 \cdot I_m$ | | | |
| $I_n \circ U_{m,k}$ | | | |
| $I_n \circ I_m$ | | | |
| $1 \circ 0 \cdot 1$ | | | |
| $1 \circ 0 \cdot I_n$ | | | |
| $1 \circ U_{n,m}$ | | | |
| $1 \circ I_n$ | | | |
| $1 \circ 1$ | | | |

TABLE 9.4. Hyperelliptic graphs, BY trees and balanced cluster pictures in genus 3

| G | T | Σ |
|------------------|-----|----------|
| Core 2 | | |
| | | |
| Core 1_n | | |
| | | |
| Core $I_{n,m}$ | | |
| | | |
| Core $U_{n,m,k}$ | | |
| | | |

TABLE 9.5. Open hyperelliptic graphs, open BY trees up to isomorphism and cluster pictures in genus 2

| G | T | Σ |
|-----------------------|-----|----------|
| Core 1×1 | | |
| | | |
| Core $1 \times I_n$ | | |
| | | |
| Core $I_n \times I_m$ | | |
| | | |

TABLE 9.5. (continued)

| Type with automorphism ϕ | G | T | Σ | pairing \langle, \rangle on Λ | action of ϕ on Λ | $ (\Lambda^\vee/\Lambda)^{\phi=1} $ |
|--------------------------------------|-----|-----|----------|--|--|--|
| 2 | | | | | | 1 |
| $1 \times_{\frac{r+s}{2}} 1$ | | | | | | 1 |
| $1 \tilde{\times}_r 1$ | | | | | | 1 |
| 1_n^+ | | | | (n) | (1) | n |
| 1_n^- | | | | (n) | (-1) | \tilde{n} |
| $1 \times_{\frac{r+s}{2}} 1_n^+$ | | | | (n) | (1) | n |
| $1 \times_{\frac{r+s}{2}} 1_n^-$ | | | | (n) | (-1) | \tilde{n} |
| $I_{n,m}^{+,+}$ | | | | $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | nm |
| $I_{n,m}^{+,-}$ | | | | $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $n\tilde{m}$ |
| $I_{n,m}^{-,-}$ | | | | $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\tilde{n}\tilde{m}$ |
| I_{n-n}^+ | | | | $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | n |
| I_{n-n}^- | | | | $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | \tilde{n} |
| $U_{n,m,k}^+$ | | | | $\begin{pmatrix} n+k & k \\ k & m+k \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | N |
| $U_{n,m,k}^-$ | | | | $\begin{pmatrix} n+k & k \\ k & m+k \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\widetilde{N/D} \times \widetilde{D}$ |
| $U_{n-n,k}^+$ | | | | $\begin{pmatrix} n+k & k \\ k & n+k \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $n+2k$ |
| $U_{n-n,k}^-$ | | | | $\begin{pmatrix} n+k & k \\ k & n+k \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ | n |
| U_{n-n-n}^+ | | | | $\begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ | 3 |
| U_{n-n-n}^- | | | | $\begin{pmatrix} 2n & n \\ n & 2n \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ | 1 |
| $I_n^+ \times_{\frac{r+s}{2}} I_m^+$ | | | | $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | nm |
| $I_n^+ \times_{\frac{r+s}{2}} I_m^-$ | | | | $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $n\tilde{m}$ |
| $I_n^- \times_{\frac{r+s}{2}} I_m^-$ | | | | $\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\tilde{n}\tilde{m}$ |
| $I_n^+ \tilde{\times}_r I_n$ | | | | $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | n |
| $I_n^- \tilde{\times}_r I_n$ | | | | $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | \tilde{n} |

Notation in the last column: $\tilde{n} = 2$ if $2|n$ and $\tilde{n} = 1$ if $2 \nmid n$; $D = \gcd(m, n, k)$; $N = nm + nk + mk$.

Black arrows in G and T and black lines in Σ indicate the automorphism; $+/-$ in T and Σ indicate the value of ϵ_ϕ . Numbers indicate lengths of edges in G and T , and distances to the parent clusters in Σ .

TABLE 9.6. Types with an automorphism in genus 2

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